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METRIC DIFFERENTIAL GEOMETRY OF RECIPROCAL RECTILINEAR CONGRUENCES[1]

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SECTION 1

INTRODUCTION

Pairs of rectilinear congruences, corresponding lines of which are polar reciprocals with respect to a chosen quadric surface, have been studied in detail by many geometers. Most of the treatment in the literature, however, has been confined to projective properties of the configurations. It is the purpose of this study to consider metric properties of reciprocal congruences, and to arrive at metric analogues of well known results in the projective group. The conventions of the tensor calculus will be observed.

In the course of the study a number of interesting geometric investigations suggest themselves, some of which are included while others are reserved for future study. Some of the results exhibited are not directly concerned with reciprocal congruences, but they are closely enough related to warrant their inclusion.

Several results are thought to be obtained here for the first time, while a number of theorems already known projectively are proved by metric methods. The quadrics of Darboux at a point of a surface are described metrically in Section 2 in a novel manner by using the definition of the quadrics to obtain conditions on the coefficients of their equation, in terms of the defining functions of the surface and of its fundamental tensors. The line of centers of the Darboux quadrics is identified through its *l-parameters*, a pair of functions of the surface coordinates used throughout the study to describe the general line of a congruence. In Section 3 the process of polar reciprocation with respect to the Darboux quadrics is used to find the vector equations of the lines of a reciprocal congruence.

The developable surfaces of a congruence and its reciprocal are studied in Section 4 by finding the differential equations in the surface coordinates of the nets of curves corresponding to the developables; in the case of the reciprocal congruence the brief equation (4.8) in general coordinates has not previously been obtained. The simultaneous conjugacy of the two nets is verified. The focal surfaces of both congruences are investigated, and are identified by vector equations.

The concept of the *normal conjugate net* of a congruence is introduced in Sec-

[1] A thesis for the Ph.D. degree, University of Oklahoma, 1947, prepared under the direction of Professor C. E. Springer.

[2] Graduate of Denison University, Class of 1930.

tion 5. It is defined there as the net of curves whose directions at each point are the direction of the normal projection upon the tangent plane to the surface at the point of the line of a congruence through the point, and the direction of its conjugate. Section 6 includes a study of the axis congruence of a net, where the axis of two curves is the common line of the osculating planes of the curves at their intersection. The l-parameters of such a congruence are obtained, with emphasis upon the special case of the axis of Čech. A new theorem is obtained which involves an expression for the angle between the axis of two curves and the normal to the surface, in terms of the curvatures of the two curves. Associate and ray congruences are studied, and a metric proof is given for the known theorem that the ray congruence of a conjugate net is identical with the reciprocal of the axis congruence of the associate net.

The interesting configuration of lines known as the canonical pencil is investigated from a metric point of view in Section 7. In the case where asymptotic curves are parametric, the l-parameters of four lines of the pencil are exhibited, viz., the projective normal, the axis of Čech, the directrix of Wilczynski, and the edge of Green. The last three of these, as well as an expression for the l-parameters of the general line of the pencil, are first obtained in this paper in metric form. A new theorem concerning a geometric relation involving three lines of the canonical pencil is given.

The last section includes a definition and some study of the *reciprocal conic*, which is the envelope of the reciprocal line of a given line at a point of a surface, as the given line generates a quadric cone with vertex at the point. It is found that the geometry of the reciprocal conic portrays geometrical properties of the line of centers of the Darboux quadrics at a point of the surface, and also that the nature of the reciprocal conic has some connection with the curvature of the surface at the point.

Definitions and Notations

The basic material for much of this work is found in Lane's treatise,[3] from which the projective work is carried over into the metric group and expressed in tensor form. The principal reference work for the fundamentals of differential geometry and the tensor calculus is that of Eisenhart,[4] whose notations are generally used herein; an exception is the use of the capital Greek gamma for the Christoffel symbol of the second kind. The summation convention is used throughout unless otherwise stated, Greek letters being summed over the range 1, 2, and Latin letters over the range 1, 2, 3. A number of notations and properties concerning the general line of a congruence have been adapted from Springer.[5]

[3] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces* (Chicago, Univ. of Chicago Press, 1932.)

[4] L. P. Eisenhart, *An Introduction to Differential Geometry with Use of the Tensor Calculus* (Princeton, Princeton University Press, 1940).

[5] C. E. Springer, *Rectilinear Congruences Whose Developables Intersect a Surface in its Lines of Curvature*, (Bulletin of the American Mathematical Society, Vol. 51, No. 12, December, 1945), p. 990. This paper is hereafter designated in footnotes by the abbreviation R. C.

SECTION 2

METRIC DESCRIPTION OF THE QUADRICS OF DARBOUX

A surface S in ordinary space is defined analytically by means of equations $x^i = x^i(u^1, u^2)$, where x^i are rectangular cartesian coordinates referred to an arbitrary origin, and $u^\alpha(\alpha = 1, 2)$ are curvilinear coordinates on the surface. It is understood that the functions x^i are continuous and have continuous derivatives of all necessary orders at every point under consideration.

A general quadric surface Q is defined in the same coordinate system by the equation

$$a_{ij}\xi^i\xi^j + 2b_i\xi^i + c = 0, \tag{2.1}$$

which contains ten coefficients. The first purpose of this section is to obtain nine equations involving the coefficients of equation (2.1), in order to arrive at a single infinity of quadrics known as the *quadrics of Darboux*. Each of the quadrics of Darboux has second order contact with S at a point P, and each enjoys a further property which will be described later. A point P with coordinates x^i on the surface S will be referred to simply as *the point* x^i.

If Q passes through x^i, then

$$c = -(a_{ij}x^ix^j + 2b_ix^i), \tag{2.2}$$

the first condition on the coefficients of equation (2.1).

The next condition imposed is that Q have second order contact with S at x^i. This requires that the first and second order terms in the series resulting when the expansion

$$\xi^i = x^i + \frac{\partial x^i}{\partial u^\alpha}\Delta u^\alpha + \frac{1}{2}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\Delta u^\alpha \Delta u^\beta + \cdots$$

is substituted into (2.1) vanish identically. The result obtained by demanding that the terms of first order vanish identically in Δu^α is

$$(a_{ij}x^j + b_i)x^i_{,\alpha} = 0. \qquad (\alpha = 1, 2)$$

This implies that

$$a_{ij}x^j + b_i = X^i, \qquad (i = 1, 2, 3) \tag{2.3}$$

where X^i are direction cosines of the normal to the surface at x^i. It may be noted that a factor of proportionality normally enters into (2.3), but the original coefficients may be chosen so that this factor is unity. For this reason it was necessary to consider (2.1) as having ten rather than nine coefficients.

Next it is required that the second order terms vanish identically. The result is

$$(a_{ij}x^j + b_i)\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + a_{ij}x^i_{,\alpha}x^j_{,\beta} = 0. \qquad \begin{pmatrix}\alpha = 1, 2\\ \beta = 1, 2\end{pmatrix} \tag{2.4}$$

By use of (2.3), equations (2.4) become

$$a_{ij}\, x^i_{,\alpha} x^j_{,\beta} + X^i \frac{\partial^2 x^i}{\partial u^\alpha\, \partial u^\beta} = 0,$$

which may be written in the form

$$a_{ij}\, x^i_{,\alpha}\, x^j_{,\beta} + d_{\alpha\beta} = 0, \qquad \begin{pmatrix}\alpha = 1,\, 2\\ \beta = 1,\, 2\end{pmatrix} \tag{2.5}$$

where $d_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface S. Equations (2.2), (2.3), (2.5) constitute seven conditions on the ten coefficients of (2.1); this agrees with the known fact that there is a triple infinity of quadrics having second order contact with S at x^i.

The directions of the curve of intersection of Q and S are given as a differential equation in the parameters u^α by equating to zero the sum of the third order terms in the expansion. If these terms are written out and reduced by use of (2.3), the result is

$$\left(a_{ij}\, x^i_{,\alpha} \frac{\partial^2 x^j}{\partial u^\beta\, \partial u^\gamma} + \tfrac{1}{3} X^i \frac{\partial^3 x^i}{\partial u^\alpha\, \partial u^\beta\, \partial u^\gamma}\right) du^\alpha\, du^\beta\, du^\gamma = 0, \tag{2.6}$$

in which increments are replaced by differentials. The second covariant derivative

$$x^j_{,\beta\gamma} \equiv \frac{\partial^2 x^j}{\partial u^\beta\, \partial u^\gamma} - x^j_{,\delta}\Gamma^\delta_{\beta\gamma} \tag{2.7}$$

is used to write (2.6) in a convenient form. By means of (2.7) the first term of (2.6) becomes

$$a_{ij} x^i_{,\alpha} x^j_{,\beta\gamma} + a_{ij}\, x^i_{,\alpha} x^j_{,\delta} \Gamma^\delta_{\beta\gamma}\, . \tag{2.8}$$

By use of (2.5) and the equations of Gauss, namely,

$$x^j_{,\,\beta\gamma} = d_{\beta\gamma} X^j$$

it is possible to write (2.8) in the form

$$a_{ij} x^i_{,\alpha} X^i d_{\beta\gamma} - d_{\alpha\delta}\, \Gamma^\delta_{\beta\gamma}\, . \tag{2.9}$$

Next, from the definition

$$d_{\beta\gamma} = X^i \frac{\partial^2 x^i}{\partial u^\beta\, \partial u^\gamma}$$

it follows that

$$\frac{\partial d_{\beta\gamma}}{\partial u^\alpha} = \frac{\partial X^i}{\partial u^\alpha} \frac{\partial^2 x^i}{\partial u^\beta\, \partial u^\gamma} + X^i \frac{\partial^3 x^i}{\partial u^\alpha\, \partial u^\beta\, \partial u^\gamma}\, . \tag{2.10}$$

Use of (2.7), the equations of Gauss, and the definition

$$d_{\alpha\delta} = -X^i_{,\alpha}\, x^i_{,\delta}$$

permit (2.10) to be written in the form

$$X^i \frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma} = \frac{\partial d_{\beta\gamma}}{\partial u^\alpha} - X^i_{,\alpha}\, d_{\beta\gamma}\, X^i + d_{\alpha\delta}\, \Gamma^\delta_{\beta\gamma}. \tag{2.11}$$

The second term in the right member of (2.11) is zero because X^i is a unit vector. Hence,

$$X^i \frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma} = \frac{\partial d_{\beta\gamma}}{\partial u^\alpha} + d_{\alpha\delta}\, \Gamma^\delta_{\beta\gamma}.$$

This expression may now be combined with (2.9) to replace (2.6) with

$$\left(3a_{ij}\, x^i_{,\alpha} X^j\, d_{\beta\gamma} + \frac{\partial d_{\beta\gamma}}{\partial u^\alpha} - 2d_{\alpha\delta}\, \Gamma^\delta_{\beta\gamma}\right) du^\alpha\, du^\beta\, du^\gamma = 0, \tag{2.12}$$

which gives the directions of the intersection of Q with S at x^i.

At each fixed point of S, the equation (2.12) defines the three directions in which the curve of intersection passes through x^i, which is always a triple point of the curve. If it happens that the three directions coincide, it is known[6] that they must coincide in one of three directions defined as the *directions of Darboux* at the point. The *quadrics of Darboux* are the quadrics each of which has second order contact with S at x^i and intersects S in a curve whose three directions at x^i are the directions of Darboux. In order to find additional conditions on the coefficients of (2.1) such that Q be a quadric of Darboux, an auxiliary condition due to Cartan[7] is introduced.

The equation (2.12) will be written for brevity in the form

$$H_{\alpha\beta\gamma}\, du^\alpha du^\beta du^\gamma = 0, \tag{2.13}$$

where

$$H_{\alpha\beta\gamma} = \frac{P}{3} M_{\alpha\beta\gamma}.$$

The symbol $M_{\alpha\beta\gamma}$ is the coefficient in (2.12) of $du^\alpha du^\beta du^\gamma$, and $\frac{P}{3} M_{\alpha\beta\gamma}$ denotes one-third of the sum of all $M_{\alpha\beta\gamma}$ whose subscripts are permutations of the same set of values for α, β, γ.

Cartan shows that the cubic form (2.12) is apolar to the quadratic form $d_{\alpha\beta} du^\alpha du^\beta = 0$, which defines the asymptotic curves on the surface, if and only if (2.12) defines the Darboux directions. Cartan's condition for apolarity of the two forms becomes in the present case

$$\begin{aligned} d_{22}H_{112} - 2d_{12}H_{122} + d_{11}H_{222} &= 0, \\ d_{22}H_{111} - 2d_{12}H_{112} + d_{11}H_{122} &= 0. \end{aligned} \tag{2.14}$$

By use of the contravariant components $d^{\alpha\beta}$ of the second surface tensor, namely,

$$d^{11} = \frac{d_{22}}{d}, \qquad d^{12} = -\frac{d_{12}}{d}, \qquad d^{22} = \frac{d_{11}}{d},$$

and by dividing out the non-zero factor d, (2.14) may be written

[6] Lane, *op. cit.*, p. 77.

[7] E. Cartan, *Leçons sur la Théorie des Espaces à Connexion Projective*, (Paris, Gauthier-Villars, 1937), p. 140.

$$d^{\beta\gamma}H_{\alpha\beta\gamma} = 0. \qquad (\alpha = 1, 2) \tag{2.15}$$

Another form of (2.12) is

$$(3a_{ij}x^i_{,\alpha}X^j d_{\beta\gamma} + 3I_{\alpha\beta\gamma})du^\alpha du^\beta du^\gamma = 0,$$

in which

$$3I_{\alpha\beta\gamma} \equiv d_{\beta\gamma,\alpha} - 2d_{\alpha\delta}\Gamma^\delta_{\beta\gamma} + d_{\beta\delta}\Gamma^\delta_{\alpha\gamma} + d_{\gamma\delta}\Gamma^\delta_{\alpha\beta}. \tag{2.16}$$

The quantities $H_{\alpha\beta\gamma}$ can be calculated in terms of $M_{\alpha\beta\gamma}$ and $I_{\alpha\beta\gamma}$, with the following results:

$$\begin{aligned} H_{111} &= M_{111} = 3a_{ij}x^i_{,1}X^j d_{11} + 3I_{111};\\ H_{112} &= H_{121} = H_{211} = \tfrac{1}{3}(M_{112} + M_{121} + M_{211})\\ &= a_{ij}X^j(x^i_{,1}\, d_{12} + x^i_{,1}\, d_{21} + x^i_{,2}\, d_{11})\\ &\quad + I_{112} + I_{121} + I_{211};\\ H_{122} &= H_{212} = H_{221} = \tfrac{1}{3}(M_{122} + M_{212} + M_{221})\\ &= a_{ij}X^j(x^i_{,1}\, d_{22} + x^i_{,2}\, d_{12} + x^i_{,2}\, d_{21})\\ &\quad + I_{122} + I_{212} + I_{221};\\ H_{222} &= M_{222} = 3a_{ij}x^i_{,2}X^j d_{22} + 3I_{222}. \end{aligned} \tag{2.17}$$

For $\alpha = 1$, (2.15) becomes

$$d^{11}H_{111} + d^{12}H_{121} + d^{21}H_{211} + d^{22}H_{221} = 0. \tag{2.18}$$

If values of $H_{\alpha\beta\gamma}$ from (2.17) are substituted into (2.18) and use is made of the identities $d^{\alpha\gamma}\, d_{\gamma\beta} = \delta^\alpha_\beta$, (2.18) assumes the form

$$\begin{aligned} 4a_{ij}x^i_{,1}X^j + 3d^{11}I_{111} + 2d^{12}(I_{112} + I_{121} + I_{211})&\\ + d^{22}(I_{122} + I_{212} + I_{221}) &= 0. \end{aligned} \tag{2.19}$$

If values of $I_{\alpha\beta\gamma}$ from (2.16) are substituted into (2.19), the result is

$$\begin{aligned} 4a_{ij}x^i_{,1}X^j + d^{11}\, d_{11,2} + \tfrac{2}{3}\, d^{12}(2d_{12,1} + d_{11,2})&\\ + \tfrac{1}{3}\, d^{22}(2d_{12,2} + d_{22,1}) &= 0. \end{aligned} \tag{2.20}$$

By the Codazzi equations, $d_{\alpha\alpha,\beta} = d_{\alpha\beta,\alpha}$, so that (2.20) reduces to

$$4a_{ij}x^i_{,1}X^j + d^{\beta\gamma}d_{\beta\gamma,1} = 0. \tag{2.21}$$

When $\alpha = 2$, (2.15) yields a similar form, which combined with (2.21) gives the final conditions

$$4a_{ij}x^i_{,\alpha}X^j + d^{\beta\gamma}\, d_{\beta\gamma,\alpha} = 0. \qquad (\alpha = 1, 2) \tag{2.22}$$

These two conditions, together with (2.2), (2.3), (2.5), provide nine conditions on the ten coefficients in the equation (2.1) of the quadric Q. The single infinity of Darboux quadrics at each point of S are now completely determined in terms of the coordinates x^i, their derivatives, and the fundamental tensors of the surface.

It may be noted here that an alternate form of (2.22) is

$$a_{ij}x^i_{,\alpha}X_j + S_\alpha = 0, \tag{2.23}$$

where S_α is defined by

$$S_\alpha \equiv \tfrac{1}{4}\, d^{\beta\gamma}\, d_{\beta\gamma,\,\alpha}\,.$$

The quantities S_α appear frequently in the sequel and are closely connected with properties of reciprocal congruences. It is not difficult to show that S_α are related to the Gaussian curvature K of the surface by the relations

$$S_\alpha = \frac{\partial}{\partial u^\alpha}\,(\log\, K^{\frac{1}{4}}). \qquad (\alpha = 1, 2) \tag{2.24}$$

The proof of (2.24) is omitted.

Lane [8] gives without proof the equation of the Darboux curves in the classical notation, for the special metric case in which the lines of curvature are taken as the coordinate curves on S. Elimination of $a_{ij}x^i_{,\alpha}\, X^j$ between (2.12) and (2.22) gives the relation

$$\left(3d^{\sigma\tau}\, d_{\beta\gamma}\, d_{\sigma\tau,\alpha} - 4\,\frac{\partial d_{\beta\gamma}}{\partial u^\alpha} + 8d_{\alpha\delta}\,\Gamma^\delta_{\beta\gamma}\right) du^\alpha\, du^\beta\, du^\gamma = 0, \tag{2.25}$$

in which the coordinate curves are any whatever. With lines of curvature parametric, $d_{12} = 0$, and (2.25) reduces to

$$\left(3d^{11}\, d_{\beta\gamma}\, d_{11,\alpha} + 3d^{22}\, d_{\beta\gamma}\, d_{22,\alpha} - 4\,\frac{\partial d_{\beta\gamma}}{\partial u^\alpha} + 8\, d_{\alpha\delta}\,\Gamma^\delta_{\beta\gamma}\right) du^\alpha\, du^\beta\, du^\gamma = 0.$$

This expression may be expanded by summing on α, β, γ to give the coefficients of the four terms of the cubic form. The coefficients may then be simplified by applying the Codazzi equations and the definitions of $\Gamma^\alpha_{\beta\gamma}$. This calculation, the details of which are omitted, leads in a straightforward way to an equation identical with Lane's except for the difference in notation.

If asymptotic curves are parametric on S, then $d_{11} = d_{22} = 0$, and (2.25) provides a second special case of the equations of the Darboux curves. The result is readily found to be

$$\Gamma^2_{11}\,(du^1)^3 + \Gamma^1_{22}\,(du^2)^3 = 0. \tag{2.26}$$

The *curves of Segre* are defined as the three curves whose directions are respectively the harmonic conjugates of the directions of Darboux with respect to the asymptotic curves. The Segre curves are evidently given, with asymptotics parametric, by the equation

$$\Gamma^2_{11}\,(du^1)^3 - \Gamma^1_{22}\,(du^2)^3 = 0. \tag{2.27}$$

The Centers of the Darboux Quadrics

In order to locate the centers of the Darboux quadrics at a point x^i of a surface S, reference is made to the observation of McConnell [9] that the center y^j of

[8] Lane, *op. cit.*, p. 240.

[9] A. J. McConnell, *Applications of the Absolute Differential Calculus* (London, Blackie and Son, Ltd., 1931), p. 114.

a quadric with the equation (2.1) will satisfy $a_{ij}y^j + b_i = 0$. Suppose that the line joining a center to the point has direction cosines λ^i. Springer[10] suggests the notation, frequently used herein,

$$\lambda^i = p^\alpha x^i_{,\alpha} + qX^i, \tag{2.28}$$

where q is the cosine of the angle θ between the normal X^i and the line of direction cosine λ^i, usually referred to simply as *the line* λ^i, and p^α are the contravariant components of a surface vector. If l^α are defined by

$$l^\alpha \equiv \frac{p^\alpha}{q},$$

it appears that a vector (not a unit vector) in the direction of λ^i is $l^\alpha x^i_{,\alpha} + X^i$. The quantities l^α will occur often in this study, and are the usual parameters by which a line passing through x^i is determined. They will be called the *l-parameters* of the line.

With the above notations, the coordinates of the center of a Darboux quadric are

$$y^j \equiv x^j + k(l^\beta x^j_{,\beta} + X^j),$$

where k is a multiplier proportional to the distance of the center from x^i. With this value of y^j, McConnell's condition is

$$a_{ij}x^j + b_i + ka_{ij}l^\beta x^j_{,\beta} + ka_{ij}X^j = 0,$$

and by use of (2.3) the last equation becomes

$$X^i + ka_{ij}l^\beta x^i_{,\beta} + ka_{ij}X^j = 0.$$

Multiplication by $x^i_{,\alpha}$, summing for i, and division by k yield

$$l^\beta a_{ij}x^i_{,\alpha}x^j_{,\beta} + a_{ij}x^i_{,\alpha}X^j = 0. \qquad (\alpha = 1, 2) \tag{2.29}$$

Upon using conditions (2.5) and (2.23) in equations (2.29), there is obtained

$$l^\beta d_{\alpha\beta} + S_\alpha = 0. \qquad (\alpha = 1, 2) \tag{2.30}$$

Since the parameters l^β determined by (2.30) are independent of the particular Darboux quadric under consideration, it appears that the centers of all Darboux quadrics at the point lie on the same line, which will be called the *line of centers*. This line can be described analytically by solving the equations (2.30) for l^β. Upon multiplication by $d^{\alpha\gamma}$ and summation for α, (2.30) becomes

$$l^\beta \delta^\gamma_\beta + S_\alpha d^{\alpha\gamma} = 0, \qquad (\gamma = 1, 2)$$

from which the desired l-parameters for the lines of centers are given by

$$l^\gamma = -S_\alpha d^{\alpha\gamma}. \qquad (\gamma = 1, 2) \tag{2.31}$$

[10] Springer, *R. C.*, p. 990.

It is readily seen that $l' = 0$ if, and only if, $S_1 = S_2 = 0$, i.e., by (2.24), K is constant, or at least the partial derivatives of K are zero at the point in question. This completes the proof of the theorem:

The centers of all Darboux quadrics at a surface point x^i lie on a line through x^i, whose direction is given by (2.31). *This line is normal to the surface if, and only if, the Gaussian curvature of the surface has zero partial derivatives at the point. It is normal at every point of a surface of constant Gaussian curvature.*

This theorem is proved otherwise in Lane.[11] Additional reference to the line of centers will be found in a later section.

Section 3

RECIPROCAL LINES AND CONGRUENCES

By the *reciprocal line* of a given line λ^i will be understood the line which is the polar reciprocal of λ^i with respect to a Darboux quadric at x^i. It will be observed that the reciprocal line of a given line λ^i is the same for all the Darboux quadrics at x^i. As the point x^i ranges over the surface S, the line λ^i generates a congruence, and the reciprocal line generates a *reciprocal congruence.*

The first problem encountered here is that of defining analytically the reciprocal of a given line λ^i. Since λ^i passes through x^i, its polar reciprocal must lie in the polar plane of x^i, which is evidently the common tangent plane of S and Q at x^i. The reciprocal line will be specified by finding the points in which it intersects the tangents to the parametric curves through x^i.

A point on the line λ^i has coordinates $x^i + k\lambda^i$. The polar plane of any point y^i with respect to Q is

$$a_{ij}\xi^i y^j + b_i(\xi^i + y^i) + c = 0.$$

Hence, the polar plane of $x^i + k\lambda^i$ is

$$a_{ij}\xi^i(x^j + k\lambda^j) + b_i(\xi^i + x^i + k\lambda^i) - a_{ij}x^ix^j - 2b_ix^i = 0,$$

where the value of c from (2.2) has been inserted. Collection of terms gives

$$(a_{ij}x^j + b_i)\xi^i + ka_{ij}\xi^i\lambda^j - (a_{ij}x^j + b_i)x^i + kb_i\lambda^i = 0,$$

which by use of (2.3) becomes

$$X^i(\xi^i + x^i) + ka_{ij}\xi^i\lambda^j + kb_i\lambda^i = 0. \tag{3.1}$$

The tangent plane to S at x^i has the equation

$$X^i(\xi^i - x^i) = 0. \tag{3.2}$$

The intersection of the planes (3.1) and (3.2) is the polar line of λ^i. By subtraction of (3.2) from (3.1) it is found that this line is also the intersection of (3.2) with the plane

$$a_{ij}\xi^i\lambda^j + b_i\lambda^i = 0. \tag{3.3}$$

[11] Lane, *op. cit.*, p. 240.

The next step is to find the points on the tangents to the coordinate curves through which the reciprocal line passes. Such a point on the tangent to the u^α-curve has coordinates

$$x^i + t_\alpha x^i_{,\alpha}, \qquad (\alpha = 1, 2) \tag{3.5}$$

where $t_\alpha(\alpha = 1, 2)$ are to be determined. In (3.5) and the remainder of this section, the subscript on t is not summed. The values of t_α, found by substituting (3.5) into (3.3), are given by

$$(a_{ij}x^j + b_i)\lambda^i + t_\alpha a_{ij}\lambda^i x^j_{,\alpha} = 0,$$

which, by use of (2.3) becomes

$$X^i\lambda^i + t_\alpha a_{ij}\lambda^i x^j_{,\alpha} = 0. \tag{3.6}$$

On substituting for λ^i from (2.28) and dividing out q, one obtains

$$1 + t_\alpha l^\sigma a_{ij}x^i_{,\sigma}x^j_{,\alpha} + t_\alpha a_{ij}X^i x^j_{,\alpha} = 0.$$

By use of (2.5) and (2.23) this becomes

$$1 - t_\alpha l^\sigma d_{\alpha\sigma} - t_\alpha S_\alpha = 0,$$

which may be solved for t_α to yield

$$t_\alpha = \frac{1}{l^\sigma d_{\alpha\sigma} + S_\alpha}. \qquad (\alpha = 1, 2) \tag{3.7}$$

It will be found useful to define the vector T_α by the relation

$$T_\alpha \equiv l^\sigma d_{\alpha\sigma} + S_\alpha . \tag{3.8}$$

It may be noted that if the line λ^i is normal to S, $l^\sigma = 0\ (\sigma = 1, 2)$. In this case, by (3.7) and (2.23), it follows that

$$t_\alpha = \frac{1}{S_\alpha} = \frac{-1}{a_{ij}\, x^i_{,\alpha}\, X_j}. \tag{3.9}$$

When the surface has constant total curvature, the centers of its Darboux quadrics are on the normal to the surface, by the theorem at the end of the previous section. The polar of the normal to S with respect to the Darboux quadrics at x^i is the line at infinity on the tangent plane to S at x^i, because the polar lies on the polar plane of the center, which is the plane at infinity. Hence, for this polar line, the quantities t_α in (3.9) are infinite, so that

$$a_{ij}x^i_{,\alpha}\, X^j = 0. \qquad (\alpha = 1, 2)$$

The reasoning is readily reversed. Therefore, we have the theorem:

A necessary and sufficient condition that a surface have constant total curvature is that the coefficients of its Darboux quadrics satisfy $a_{ij}x^i_{,\alpha}\, X^j = 0$ at every point of the surface.

Section 4

INTERSECTOR AND RECIPROCAL NETS, FOCAL SURFACES

With the analytic framework provided in Section 3, it is possible to investigate other aspects of a congruence and its reciprocal, relative to a given surface. Of these, perhaps the most important is the determination of the developable surfaces and focal surfaces of the congruences. It is known that the lines of a congruence can be assembled into two families of developable surfaces, which meet the given surface S in two families of curves called the *intersector net* of the congruence. The developables are identified by finding the differential equation of the intersector net in the surface coordinates. Upon each line are two focal points, which are the points at which the line is tangent to the edges of regression of the developables containing the line. As the line ranges over the congruence, these points generate the focal surfaces.

For the developables of the reciprocal congruence a slightly different concept is used. As a reciprocal line generates a developable in its congruence, the corresponding point x^i traces a curve on the surface. The net on S of such curves corresponding to the two one-parameter families of developables in the reciprocal congruence will be called the *reciprocal net*. It should be noted that this is not the same as the intersector net of the reciprocal congruence.

The Intersector Net

For the congruence generated by a line of direction cosines λ^i through a point x^i of the surface, Springer[12] gives the equation of the intersector net as

$$e_{\alpha\gamma}(l^\gamma \nu_\beta - \mu^\gamma_\beta)\, du^\alpha\, du^\beta = 0, \tag{4.1}$$

where

$$\begin{aligned} \nu_\beta &= q_{,\beta} + p^\sigma d_{\beta\sigma}, \\ \mu^\gamma_\beta &= p^\gamma_{,\beta} - q d_{\beta\sigma} g^{\sigma\gamma}, \end{aligned} \tag{4.2}$$

and where $e_{\alpha\gamma}$ is defined by

$$e_{11} = e_{22} = 0, \qquad e_{12} = 1, \qquad e_{21} = -1.$$

By replacing ν_β and μ^γ_β by their values from (4.2), and using $p^\gamma = ql^\gamma$, (4.1) can be written

$$e_{\alpha\sigma}(l^\sigma_{,\beta} - d_{\beta\gamma} g^{\sigma\gamma} - l^\sigma l^\lambda d_{\lambda\beta})\, du^\alpha du^\beta = 0, \tag{4.3}$$

for which a later use will be found.

The Reciprocal Net

To find the reciprocal net, several methods of attack are available. The one which has led most satisfactorily to the desired result is used by Green[13] in the

[12] Springer, *R.C.*, p. 992.

[13] G. M. Green, *Memoir on the General Theory of Surfaces and Rectilinear Congruences* (Transactions of the American Mathematical Society, Vol. 20, 1919), p. 88.

projective case. A developable surface has the property that the tangent planes at any two points on a generator are the same. Hence any two points ξ^i and η^i on a generator are coplanar with the points $\xi^i + d\xi^i$ and $\eta^i + d\eta^i$ into which they are carried by a slight displacement. Thus if x^i moves along a curve such that the line associated with x^i in the congruence under investigation generates a developable surface, it follows that the determinant

$$(4.4) \qquad \begin{vmatrix} \xi^1 & \xi^2 & \xi^3 & 1 \\ \eta^1 & \eta^2 & \eta^3 & 1 \\ \xi^1 + d\xi^1 & \xi^2 + d\xi^2 & \xi^3 + d\xi^3 & 1 \\ \eta^1 + d\eta^1 & \eta^2 + d\eta^2 & \eta^3 + d\eta^3 & 1 \end{vmatrix}$$

must vanish. If applied to the congruence of lines λ^i, this method yields the equation (4.1) without difficulty.

In the case of the reciprocal congruence, one may use for the two points those in which the generator meets the parametric tangents through the associated point x^i, namely,

$$\xi^i = x^i + t_1 x^i_{,1}, \qquad \eta^i = x^i + t_2 x^i_{,2},$$

where t_α are given by (3.7). If these values are substituted into (4.4), and obvious reductions are made, the condition appears in the form

$$\begin{vmatrix} t_2 x^i_{,2} - t_1 x^i_{,1} \\ dx^j + t_1\, dx^j_{,1} + x^j_{,1}\, dt_1 \\ dx^k + t_2\, dx^k_{,2} + x^k_{,2}\, dt_2 \end{vmatrix} = 0,$$

where the three columns are formed by letting i, j, k take values 1, 2, 3. This may be written in the form

$$(4.5) \qquad \delta^{123}_{ijk}(t_2 x^i_{,2} - t_1 x^i_{,1})(dx^j + t_1 dx^j_{,1} + x^j_{,1}\, dt_1)(dx^k + t_2\, dx^k_{,2} + x^k_{,2}\, dt_2) = 0.$$

The differentials dx^j in (4.5) may be replaced by $x^j_{,\alpha}\, du^\alpha$. The determinant is expanded into a sum of determinants, several of which vanish because two rows are identical. The second partial derivatives are written in terms of the covariant derivatives of x^i, and the Gauss equations are applied to make possible the substitution

$$(4.6) \qquad \frac{\partial^2 x^i}{\partial u^\alpha\, \partial u^\beta} = d_{\alpha\beta}\, X^i + x^i_{,\delta}\Gamma^\delta_{\alpha\beta}.$$

Certain terms are found to be zero, and those remaining have the common factor

$$\delta^{123}_{ijk}\, x^i_{,1}\, x^j_{,2}\, X^k,$$

which is not zero (since the vectors $x^i_{,1}$, $x^i_{,2}$, X^i are not coplanar) and can be divided out. Thus, (4.5) leads to the following form of the equation of the reciprocal net:

$$t_1^2 d_{1\beta}\, du^2 du^\beta - t_2^2 d_{2\beta}\, du^1 du^\beta + t_1 t_2 (d_{1\beta}\, du^1 du^\beta - d_{2\beta}\, du^2 du^\beta)$$

$$
\begin{aligned}
&+ t_1^2 t_{2,\alpha} d_{1\beta}\, du^\alpha du^\beta - t_2^2 t_{1,\alpha} d_{2\beta}\, du^\alpha du^\beta \\
&+ [t_1^2 t_2 (d_{1\alpha}\Gamma_{2\beta}^2 - d_{2\alpha}\Gamma_{1\beta}^2) - t_2^2 t_1 (d_{2\alpha}\Gamma_{1\beta}^1 - d_{1\alpha}\Gamma_{2\beta}^1)]\, du^\alpha du^\beta = 0.
\end{aligned} \tag{4.7}
$$

It may be remarked that from equation (4.7) one may verify the form given by Lane for the particular case in which the lines of curvature are coordinate.[14]

If each term of (4.7) is divided by $t_1^2 t_2^2$, and if $\dfrac{1}{t_1}$ and $\dfrac{1}{t_2}$ are replaced by the vectors T_1 and T_2 defined in (3.8), the result can be written in the compact form

$$
e^{\gamma\delta} d_{\alpha\gamma} (T_\delta T_\beta - T_{\delta,\beta})\, du^\alpha du^\beta = 0. \tag{4.8}
$$

This is the differential equaticn of the reciprocal net. Another useful form, obtained by replacing T_α by their values from (3.8), is

$$
\begin{aligned}
(d e_{\alpha\sigma} l^\sigma l^\lambda d_{\lambda\beta} - d e_{\alpha\sigma} l^\sigma_{,\beta})\, du^\alpha du^\beta + e^{\gamma\delta} d_{\alpha\gamma} (d_{\sigma\delta} l^\sigma S_\beta + d_{\sigma\beta} l^\sigma S_\delta \\
+ S_\delta S_\beta - l^\sigma d_{\sigma\delta,\beta} - S_{\delta,\beta})\, du^\alpha du^\beta = 0.
\end{aligned} \tag{4.9}
$$

The condition for any net $A_{\alpha\beta}\, du^\alpha du^\beta = 0$ to be a conjugate net is that $A_{\alpha\beta}\, d^{\alpha\beta} = 0$. Applying this to the third term in the equation (4.3) of the intersector net gives

$$
e_{\alpha\sigma} l^\sigma l^\lambda d_{\lambda\beta}\, d^{\alpha\beta} = e_{\alpha\sigma} l^\sigma l^\lambda \delta^\alpha_\lambda = e_{\alpha\sigma} l^\sigma l^\alpha = 0.
$$

A comparison of the form of the third term of (4.3) with the first shows that the latter will also vanish. Hence the conjugacy condition reduces to

$$
e_{\alpha\sigma} l^\sigma_{,\beta}\, d^{\alpha\beta} = 0, \tag{4.10}
$$

a result obtained otherwise by Springer.[15]

If the same condition is applied to the reciprocal net (4.9), it is observed that the terms of the first parentheses were treated in the preceding paragraph. The expression in the second parentheses is symmetric in β and δ, and may be written $B_{\beta\delta}$. This condition gives

$$
e^{\gamma\delta} d_{\alpha\gamma} B_{\beta\delta}\, d^{\alpha\beta} = e^{\gamma\delta} B_{\beta\delta} \delta^\beta_\gamma = e^{\beta\delta} B_{\beta\delta} = 0.
$$

Hence the conjugacy conditiȯn for the whole of (4.9) reduces again to (4.10). This proves the theorem, stated by Lane,[16]

The intersector net is conjugate if, and only if, the reciprocal net is conjugate.

The Focal Surfaces

Of the possible methods of finding focal points, and hence the surfaces generated by them, the method of Green [17] has been chosen because it parallels the technique used in obtaining the equations of the intersector and reciprocal nets.

[14] Lane, *op. cit.*, p. 253.
[15] Springer, *R.C.*, p. 993.
[16] Lane, *op. cit.*, p. 87.
[17] Green, *op. cit.*, p. 90.

If $\xi^i = x^i + t\lambda^i$ is a focal point on λ^i, it generates a surface. The points $\xi^i + \dfrac{\partial \xi^i}{\partial u^1}$ and $\xi^i + \dfrac{\partial \xi^i}{\partial u^2}$, as well as x^i and $x^i + \lambda^i$, are in the tangent plane to that surface at x^i. Hence

$$\begin{vmatrix} x^1 & x^2 & x^3 & 1 \\ x^1 + \lambda^1 & \cdots & \cdots & 1 \\ x^1 + t\lambda^1 + x^1_{,1} + t\lambda^1_{,1} + \lambda^1 t_{,1} & \cdots & \cdots & 1 \\ x^1 + t\lambda^1 + x^1_{,2} + t\lambda^1_{,2} + \lambda^1 t_{,2} & \cdots & \cdots & 1 \end{vmatrix} = 0. \tag{4.11}$$

By simple reduction, equation (4.11) leads to the equation

$$\delta^{123}_{ijk}\lambda^i(x^j_{,1} + t\lambda^j_{,1})(x^k_{,2} + t\lambda^k_{,2}) = 0, \tag{4.12}$$

a quadratic in t, the roots of which give the focal points of the line λ^i. If λ^i and its derivatives are replaced by their values in terms of $x^i_{,\alpha}$ and X^i, a reduction of the same type as that employed in finding the reciprocal net leads to the equation

$$\begin{aligned}(l^1\mu^2_1\nu_2 - l^2\mu^1_1\nu_2 - l^1\mu^2_2\nu_1 + l^2\mu^1_2\nu_1 + \mu^1_1\mu^2_2 \\ - \mu^2_1\mu^1_2)t^2 + (\mu^2_2 + \mu^1_1 - l^2\nu_2 - l^1\nu_1)t + 1 = 0.\end{aligned} \tag{4.13}$$

If h^γ_β is written in place of $\mu^\gamma_\beta - l^\gamma\nu_\beta$, an expression which appears both here and in the intersector net, (4.13) can be written in the form

$$(h^1_1h^2_2 - h^2_1h^1_2)t^2 + (h^1_1 + h^2_2)t + 1 = 0. \tag{4.14}$$

This equation does not lend itself readily to writing in tensor form. If the lines of curvature are taken as coordinate, and (4.13) is correspondingly evaluated, the result agrees with the form given by Lane[18] for this case.

If the lines λ^i of the congruence are the normals to the surface, then $l^\alpha = 0$, $q = 1$, and (4.13) becomes

$$(\mu^1_1\mu^2_2 - \mu^2_1\mu^1_2)t^2 + (\mu^2_2 + \mu^1_1)t + 1 = 0, \tag{4.15}$$

where

$$\mu^1_1 = -d_{11}g^{11} - d_{12}g^{12} = \frac{-d_{11}g_{22} + d_{12}g_{12}}{g}$$

and similarly for μ^1_2, μ^2_1, μ^2_2. Using these quantities in (4.15) and multiplying by the determinant g, one finds

$$dt^2 + (2d_{12}g_{12} - d_{11}g_{22} - d_{22}g_{11})t + g = 0. \tag{4.16}$$

This form is given by Eisenhart.[19] The roots of (4.16) are ρ_1, ρ_2, the principal radii of normal curvature of S at x^i.

[18] Lane, *op. cit.*, p. 252.
[19] Eisenhart, *op. cit.*, p. 225.

A similar procedure yields the focal points for the reciprocal congruence, though the process is longer because of the more complicated expressions involved. Here the general point on the reciprocal line is

$$\xi^i = x^i + rt_1 x^i_{,1} + (1 - r)t_2 x^i_{,2}\,.$$

The four coplanar points are

$$\xi^i + \frac{\partial \xi^i}{\partial u^1}\,;\, \xi^i + \frac{\partial \xi^i}{\partial u^2}\,;\, x^i + t_1 x^i_{,1}\,;\, x^i + t_2 x^i_{,2}\,.$$

The fourth order determinant is simplified, expanded, and reduced as before, and the vectors T_α introduced as in the reciprocal net. The result is a quadratic equation in r, the roots of which give the focal points. The equation may be written briefly by introducing the quantities

$$Q_1 \equiv rT_2 \quad ; \quad Q_2 \equiv (1 - r)T_1\,.$$

After this manipulation the equation appears in the form

$$e^{\gamma\delta}\, d_{\alpha\gamma}(T_{\beta,\delta} - T_\beta T_\delta)Q_\alpha Q_\beta = 0, \tag{4.17}$$

which is analogous in form to the equation (4.8) of the reciprocal net. In the case where the lines of curvature are coordinate, equation (4.17) can be shown to agree with the result given by Lane.[20]

If a root of (4.14) is inserted for t in the form $\xi^i = x^i + t\lambda^i$, then the latter is the vector equation of the corresponding focal surface of the given congruence. A similar statement holds for the equations of the focal surfaces of the reciprocal congruence.

Section 5

THE NORMAL CONJUGATE NET OF A CONGRUENCE

Any two distinct lines $\lambda^i_{(1)}$ and $\lambda^i_{(2)}$ through x^i and not in the tangent plane to S at x^i determine a plane. Its intersection with the tangent plane to S at x^i is a tangent line whose direction, as a vector, is given by a linear combination of the given vectors $l^\sigma_{(1)}x^i_{,\sigma} + X^i$ and $l^\sigma_{(2)}x^i_{,\sigma} + X^i$, and simultaneously of the coordinate tangent vectors $x^i_{,1}$ and $x^i_{,2}$. A vector satisfying these conditions is seen by inspection to be

$$(l^\sigma_{(1)} - l^\sigma_{(2)})x^i_{,\sigma}\,. \tag{5.1}$$

The polar point of the given plane with respect to a Darboux quadric at x^i is the intersection of the reciprocals of $\lambda^i_{(1)}$ and $\lambda^i_{(2)}$. To find this point, one may write the coordinates of the general points ξ^i and η^i on the reciprocals of $\lambda^i_{(1)}$ and $\lambda^i_{(2)}$ in the respective forms

$$\begin{aligned} \xi^i &= x^i + r\left(\frac{1}{T^{(1)}_1}\right)x^i_{,1} + (1 - r)\left(\frac{1}{T^{(1)}_2}\right)x^i_{,2}, \\ \eta^i &= x^i + t\left(\frac{1}{T^{(2)}_1}\right)x^i_{,1} + (1 - t)\left(\frac{1}{T^{(2)}_2}\right)x^i_{,2}. \end{aligned} \tag{5.2}$$

[20] Lane, *op. cit.*, p. 253.

The points represented in (5.2) coincide if, and only if,

$$\frac{r}{T_1^{(1)}} = \frac{t}{T_1^{(2)}}; \qquad \frac{1-r}{T_2^{(1)}} = \frac{1-t}{T_2^{(2)}}.$$

By eliminating r and solving for t, it is found that the required point has the coordinates

$$(5.3) \qquad \eta^i = x^i + \frac{1}{T_1^{(2)}T_2^{(1)} - T_2^{(2)}T_1^{(1)}}\left[(T_2^{(1)} - T_2^{(2)})x^i_{,1} - (T_1^{(1)} - T_1^{(2)})x^i_{,2}\right].$$

From (3.8) it is seen that

$$T_\alpha^{(1)} - T_\alpha^{(2)} = (l^\sigma_{(1)} - l^\sigma_{(2)})d_{\sigma\alpha}. \qquad (\alpha = 1, 2)$$

Hence the direction of the tangent line joining x^i and η^i is given by

$$(5.4) \qquad (l^\sigma_{(1)} - l^\sigma_{(2)})d_{\sigma 2}x^i_{,1} - (l^\sigma_{(1)} - l^\sigma_{(2)})d_{\sigma 1}x^i_{,2}.$$

In case the coordinate curves are the asymptotics, (5.4) becomes

$$(5.5) \qquad d_{12}[(l^1_{(1)} - l^1_{(2)})x^i_{,1} - (l^2_{(1)} - l^2_{(2)})x^i_{,2}].$$

This direction is conjugate to that of the first tangent, given by (5.1). Furthermore, the two tangents so defined are polar reciprocals. This is seen by observing that the first tangent is the intersection of the tangent plane and the plane of $\lambda^i_{(1)}$ and $\lambda^i_{(2)}$; its reciprocal is therefore the line joining the polar points of those planes, namely, x^i and η^i. If now one takes any tangent through x^i and locates its reciprocal, the reasoning above shows that reciprocal to be the conjugate tangent. This establishes the theorem

Pairs of conjugate lines through x^i in the tangent plane are pairs of reciprocal lines.

If one of the given lines $\lambda^i_{(1)}$, $\lambda^i_{(2)}$ is the normal X^i, and the other is the corresponding line of a given congruence, the two tangents determined as above may be called the *normal projection tangent* and the *reciprocal tangent* of the congruence at x^i. As x^i moves over the surface and λ^i generates the congruence, the directions of the two tangents are always conjugate; the net of conjugate directions so determined may be called the *normal conjugate net* of the congruence. The direction of the normal projection tangent is found by putting $l^\sigma_{(1)} = l^\sigma$ and $l^\sigma_{(2)} = 0$ in (5.1), yielding the direction

$$(5.6) \qquad l^1x^i_{,1} + l^2x^i_{,2}.$$

Similarly the direction of the reciprocal tangent is given by

$$(5.7) \qquad l^1x^i_{,1} - l^2x^i_{,2}.$$

Here and in the remainder of this section, the asymptotic curves are taken as parametric, because this choice is best adapted to the discussion of conjugate directions.

The directions (5.6) and (5.7) may be expressed in terms of the equation of a net instead of in vector form. In general, if a direction is given by $A^\alpha x^i_{,\alpha}$,

one may assume a small displacement dx^i in the direction of the vector. Then, with t a factor of proportionality, one has the relations

$$A^\alpha x^i_{,\alpha} = t dx^i = t x^i_{,\alpha} du^\alpha, \qquad (A^\alpha - t du^\alpha) x^i_{,\alpha} = 0,$$

whence

$$A^1 - t du^1 = 0, \qquad A^2 - t du^2 = 0,$$

and by eliminating t,

$$A^2 du^1 - A^1 du^2 = 0. \tag{5.8}$$

The direction given by (5.8) is the same as that of the given vector $A^\alpha x^i_{,\alpha}$. Two directions give rise to the equation of a net. From (5.6), (5.7), (5.8) it follows that the equation of the normal conjugate net is

$$(l^2)^2 (du^1)^2 - (l^1)^2 (du^2)^2 = 0. \tag{5.9}$$

The intersector net (4.3), when asymptotics are parametric, becomes

$$\begin{aligned}[l^2_{,1} - d_{12} g^{22} - (l^2)^2 d_{12}](du^1)^2 + [l^2_{,2} - l^1_{,1}] du^1 du^2 \\ +(-l^1_{,2} + d_{12} g^{11} + (l^1)^2 d_{12}](du^2)^2 = 0.\end{aligned} \tag{5.10}$$

If (5.9) is multiplied by $-d_{12}$ and compared with (5.10), the conditions for the two nets to be identical are seen to be

$$l^2_{,1} = d_{12} g^{22}; \qquad l^2_{,2} - l^1_{,1} = 0; \qquad l^1_{,2} = d_{12} g^{11},$$

which may be written

$$l^\sigma_{,\tau} = d_{\tau\rho} g^{\rho\sigma}. \qquad (\sigma, \tau = 1, 2) \tag{5.11}$$

The l-parameters of a congruence, the intersector net of which coincides with the normal conjugate net, must satisfy the partial differential equations (5.11).

One may find the congruence whose normal conjugate net coincides with the lines of curvature net. The latter, with asymptotics parametric, has the equation

$$g_{11} (du^1)^2 - g_{22} (du^2)^2 = 0. \tag{5.12}$$

Comparison of (5.12) with (5.9) shows that the two nets are identical if, and only if,

$$(l^1)^2 = k g_{22}, \qquad (l^2)^2 = k g_{11}.$$

Since k is arbitrary, the result can be stated in the theorem

The congruence whose l-parameters are respectively $\sqrt{g_{22}}$ and $\sqrt{g_{11}}$ (asymptotics being parametric) has for its normal conjugate net the lines of curvature.

Section 6

AXES AND AXIS CONGRUENCES

The *axis* of two curves on a surface S is defined as the line of intersection of their osculating planes at the point x^i where the curves meet. The two given curves C_1, C_2 are defined by

$$u^\alpha = u^\alpha(S_{(1)}), \qquad u^\alpha = u^\alpha(S_{(2)}),$$

where $S_{(1)}$ and $S_{(2)}$ represent arc-length along C_1 and C_2 respectively. The equation

$$(6.1)\qquad e_{\sigma\tau}u'^{\sigma}(\rho^{\tau} - \kappa_n l^{\tau}) = 0, \qquad \left(u' \equiv \frac{du}{ds}\right),$$

given by Springer,[21] expresses the fact that the line of direction cosines λ^i lies in the osculating plane of the curve $C: u^{\alpha} = u^{\alpha}(S)$ at x^i; ρ^{τ} are the contravariant components of the curvature vector of C at x^i, and κ_n is the normal curvature of C at x^i.

If l^{τ} are the parameters for the axis of C_1 and C_2, the following two equations hold simultaneously, from (6.1):

$$(6.2)\qquad \begin{aligned} u'^2_{(1)} l^1 - u'^1_{(1)} l^2 &= \frac{1}{\kappa_n^{(1)}} (u'^2_{(1)} \rho^1_{(1)} - u'^1_{(1)} \rho^2_{(1)}), \\ u'^2_{(2)} l^1 - u'^1_{(2)} l^2 &= \frac{1}{\kappa_n^{(2)}} (u'^2_{(2)} \rho^1_{(2)} - u'^1_{(2)} \rho^2_{(2)}). \end{aligned}$$

The linear equations (6.2) may be solved for l^1 and l^2, provided that $\kappa_n^{(1)}\kappa_n^{(2)} \neq 0$ and $u'^1_{(1)} u'^2_{(2)} - u'^1_{(2)} u'^2_{(1)} \neq 0$, which conditions can be interpreted geometrically as follows. If either of the normal curvatures is zero, the corresponding curve is an asymptotic. Its osculating plane is the tangent plane, and the axis lies in the tangent plane, a case excluded here. The expression $u'^1_{(1)} u'^2_{(2)} - u'^1_{(2)} u'^2_{(1)}$ is equal to[22] $\frac{1}{\sqrt{g}} \sin\psi$, where ψ is the angle between C_1 and C_2. Since C_1 and C_2 are distinct, the angle ψ is different from zero.

Equations (6.2) yield the solutions

$$(6.3)\qquad l^{\alpha} = \frac{\sqrt{g}}{\sin\psi} e^{\gamma\delta} e_{\sigma\tau} u'^{\alpha}_{(\gamma)} \left(\frac{1}{\kappa_n} \rho^{\sigma} u'^{\tau}\right)_{(\delta)}, \qquad (\alpha = 1, 2)$$

An alternative form for equations (6.3) is obtained by using geodesic curvature of a curve C at x^i, defined by[23]

$$\kappa_g = \sqrt{g}\, e_{\sigma\tau} u'^{\sigma} \rho^{\tau},$$

which leads to

$$(6.4)\qquad l^{\alpha} = \frac{1}{\sin\psi} e^{\gamma\delta} u'^{\alpha}_{(\gamma)} \left(\frac{\kappa_g}{\kappa_n}\right)_{(\delta)}.$$

Still another form is obtained from the relations[24]

$$\kappa_n = \frac{\cos\bar{\omega}}{\rho}, \qquad \kappa_g = \frac{\sin\bar{\omega}}{\rho}.$$

[21] C. E. Springer, *Union Curves and Union Curvature* (Bulletin of the American Mathematical Society, Vol. 51, No. 10, October, 1945), p. 688. This paper is designated in footnotes by the abbreviation U.C.

[22] Eisenhart, *op. cit.*, p. 131.

[23] Eisenhart, *op. cit.*, p. 187.

[24] Eisenhart, *op. cit.*, pp. 224, 246.

where ρ is the ordinary radius of curvature of a curve C, and $\bar{\omega}$ is the angle between the normal to the surface and the principal normal to the curve at x^i. It follows that

$$\frac{\kappa_g}{\kappa_n} = \tan \bar{\omega},$$

whence (6.4) takes the form

$$l^\alpha = \frac{1}{\sin \psi} e^{\gamma\delta} u'^\alpha_{(\gamma)} \tan \bar{\omega}_{(\delta)}. \tag{6.5}$$

If C_1 and C_2 are both geodesics, the curvature vector is a zero vector for each, and from (6.3) the axis is the normal to the surface. This, of course, is to be expected, since in this case both osculating planes contain the normal.

If the curves C_α are the coordinate curves, Springer[25] shows that the l-parameters of the axis are

$$l^1 = \frac{1}{d_{22}} \Gamma^1_{22}, \qquad l^2 = \frac{1}{d_{11}} \Gamma^2_{11}. \tag{6.6}$$

From the foregoing results there follow some interesting geometric consequences. The relation

$$\lambda^i \lambda^i = 1 = g_{\alpha\beta} l^\alpha l^\beta q^2 + q^2, \tag{6.7}$$

follows from (2.28), where it is recalled that $q = \cos \theta$, θ being the angle between the line λ^i and the normal. Another form of (6.7) is

$$\tan^2 \theta = g_{\alpha\beta} l^\alpha l^\beta. \tag{6.8}$$

It is of interest to determine θ for the axis of two curves C_α. The l-parameters given by (6.4) may be written in the form

$$l^\alpha = \frac{1}{\sin \psi} e^{\gamma\delta} u'^\alpha_{(\gamma)} (\kappa_g R_n)_{(\delta)}. \qquad (\alpha = 1, 2) \tag{6.9}$$

where $R_n(\delta)$ is the radius of normal curvature for the curve C_δ at x^i. Substitution from (6.9) into (6.8) yields

$$\tan^2 \theta = \frac{1}{\sin^2 \psi} g_{\alpha\beta} e^{\gamma\delta} u'^\alpha_{(\gamma)} (\kappa_g R_n)_{(\delta)} e^{\sigma\tau} u'^\beta_{(\sigma)} (\kappa_g R_n)_{(\tau)}. \tag{6.10}$$

Expansion of (6.10) leads to the relation

$$\tan^2 \theta = \frac{1}{\sin^2 \psi} [(\kappa_{g(2)} R_{n(2)})^2 g_{\alpha\beta} u'^\alpha_{(1)} u'^\beta_{(1)} + (\kappa_{g(1)} R_{n(1)})^2 g_{\alpha\beta} u'^\alpha_{(2)} u'^\beta_{(2)} - 2(\kappa_{g(1)} R_{n(1)} \kappa_{g(2)} R_{n(2)} g_{\alpha\beta} u'^\alpha_{(1)} u'^\beta_{(2)}],$$

whence there follows[26] the form

[25] Springer, *U.C.*, p. 690.
[26] Eisenhart, *op. cit.*, p. 130.

$$\tan^2\theta = \frac{1}{\sin^2\psi}[(\kappa_{g(1)}R_{n(1)})^2 + (\kappa_{g(2)}R_{n(2)})^2 - 2\kappa_{g(1)}R_{n(1)}\kappa_{g(2)}R_{n(2)}\cos\psi]. \tag{6.11}$$

If the curves are orthogonal, (6.11) reduces to

$$\tan^2\theta = (\kappa_{g(1)}R_{n(1)})^2 + (\kappa_{g(2)}(R_{n(2)})^2. \tag{6.12}$$

The result (6.12) is expressed in the theorem

The square of the tangent of the angle between the axis of two orthogonal curves on a surface and the normal to the surface at a point is equal to the sum of the squares of the products of the geodesic curvatures by the corresponding radii of normal curvature of the two curves at the point.

The similarity of (6.11) to the Law of Cosines suggests the following geometric relation: if segments are laid off on the tangents to two curves at a point, equal in length respectively to the products of the geodesic curvatures by the corresponding radii of normal curvature of the curves at the point, the length of the third side of the triangle so formed is equal to the product of the sine of the angle between the curves by the tangent of the angle between their axis and the normal. In the case of orthogonal curves, the length of the third side is equal to the tangent of the latter angle.

The Axis of Čech

It is well known in the projective theory that the osculating planes of the three curves of Segre at a point have a line in common, known as the *axis of Čech.* In this sub-section the l-parameters of the axis of Čech are exhibited. The asymptotics are used as coordinate curves.

The equation (6.1) may be expanded, making use of the relations

$$\rho^\tau = u''^\tau + \Gamma^\tau_{\gamma\delta}u'^\gamma u'^\delta,$$

$$\kappa_n = d_{\gamma\delta}u'^\gamma u'^\delta,$$

$$\frac{d^2v}{ds^2} = \frac{d^2v}{du^2}\left(\frac{du}{ds}\right)^2 + \frac{dv}{du}\frac{d^2u}{ds^2},$$

where for convenience u and v are used in place of u^1 and u^2. The result is found to be

$$\begin{aligned}\frac{d^2v}{du^2} + \Gamma^2_{11} + 2\Gamma^2_{12}\frac{dv}{du} + \Gamma^2_{22}\left(\frac{dv}{du}\right)^2 - 2d_{12}\frac{dv}{du}l^2 \\ - \Gamma^1_{11}\frac{dv}{du} - 2\Gamma^1_{12}\left(\frac{dv}{du}\right)^2 - \Gamma^1_{22}\left(\frac{dv}{du}\right)^3 + 2d_{12}\left(\frac{dv}{du}\right)^2 l^1 = 0.\end{aligned} \tag{6.13}$$

The curves of Segre, denoted here by C_1, C_2, and C_3, are represented by equation (2.27), which may be written in the form

$$\left(\frac{dv}{du}\right)^3 = r^3, \tag{6.14}$$

where

$$r = \left(\frac{\Gamma^2_{11}}{\Gamma^1_{22}}\right)^{\frac{1}{3}}.$$

From (6.14) it is seen that the following relations hold.

$$C_1: \quad \frac{dv}{du} = r, \qquad \frac{d^2v}{du^2} = r\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u},$$

$$C_2: \quad \frac{dv}{du} = \omega r, \qquad \frac{d^2v}{du^2} = \omega^2 r\frac{\partial r}{\partial v} + \omega\frac{\partial r}{\partial u},$$

$$C_3: \quad \frac{dv}{du} = \omega^2 r, \qquad \frac{d^2v}{du^2} = \omega r\frac{\partial r}{\partial v} + \omega^2\frac{\partial r}{\partial u},$$

where ω is the complex cube root of unity. It follows that (6.13) for the curve C_1 becomes

$$(6.15) \qquad r\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u} + 2r\Gamma^2_{12} + r^2\Gamma^2_{22} - 2rd_{12}l^2 - r\Gamma^1_{11} - 2r^2\Gamma^1_{12} + 2r^2 d_{12}l^1 = 0.$$

Similar equations hold for C_2 and C_3, if r in (6.15) is replaced by ωr and $\omega^2 r$ respectively. The three equations so obtained are linearly dependent, so that a solution of two of them is a solution of the third. The l-parameters for the axis of Čech may therefore be found by solving the first two for l^1 and l^2. The solutions are

$$l^1 = \frac{\Gamma^1_{12}}{d_{12}} - \frac{\Gamma^2_{22}}{2d_{12}} - \frac{1}{2d_{12}}\frac{\partial}{\partial v}\log r,$$

$$l^2 = \frac{\Gamma^2_{12}}{d_{12}} - \frac{\Gamma^1_{11}}{2d_{12}} + \frac{1}{2d_{12}}\frac{\partial}{\partial u}\log r.$$

By inserting the value of r in terms of Christoffel symbols as given in (6.14), a more useful form is obtained, namely,

$$(6.16) \qquad \begin{aligned} l^1 &= \frac{1}{2d_{12}}\left(\tfrac{1}{3}\frac{\partial}{\partial u^2}\log\Gamma^1_{22} - \tfrac{1}{3}\frac{\partial}{\partial u^2}\log\Gamma^2_{11} + 2\Gamma^1_{12} - \Gamma^2_{22}\right), \\ l^2 &= \frac{1}{2d_{12}}\left(\tfrac{1}{3}\frac{\partial}{\partial u^1}\log\Gamma^2_{11} - \tfrac{1}{3}\frac{\partial}{\partial u^1}\log\Gamma^1_{22} + 2\Gamma^2_{12} - \Gamma^1_{11}\right). \end{aligned}$$

Axis and Ray Congruences of a Conjugate Net

In this sub-section there is developed the metric analogue of the projective theory of the axis and ray congruences of a conjugate net.[27] The asymptotic curves are taken as the coordinate curves.

If the radius of normal curvature R_n of a curve C is introduced in place of the normal curvature κ_n, (6.2) may be written in the form

$$(6.17) \qquad \begin{aligned} \frac{du^2_{(1)}}{du^1_{(1)}}l^1 - l^2 &= R^{(1)}_n\left(\frac{du^2_{(1)}}{du^1_{(1)}}\overset{1}{\rho}_{(1)} - \overset{2}{\rho}_{(1)}\right), \\ \frac{du^2_{(2)}}{du^1_{(2)}}l^1 - l^2 &= R^{(2)}_n\left(\frac{du^2_{(2)}}{du^1_{(2)}}\overset{1}{\rho}_{(2)} - \overset{2}{\rho}_{(2)}\right). \end{aligned}$$

[27] Lane, *op. cit.*, pp. 95 ff.

If C_1 and C_2 are the two curves of a conjugate net passing through x^i, where the net is given by the equation $k(du^1)^2 - (du^2)^2 = 0$, then (6.17) becomes

$$
\begin{aligned}
kl^1 - l^2 &= R_n^{(1)}(k\rho_{(1)}^1 - \rho_{(1)}^2), \\
kl^1 + l^2 &= R_n^{(2)}(k\rho_{(2)}^1 + \rho_{(2)}^2).
\end{aligned} \tag{6.18}
$$

After introduction of the values of R_n and ρ^α as in the preceding sub-section, the solutions of (6.18) are found to be

$$
\begin{aligned}
l^1 &= \frac{1}{2k^2 d_{12}}\left(2k^2\Gamma_{12}^1 - \Gamma_{11}^2 - k^2\Gamma_{22}^2 - k\frac{\partial k}{\partial u^2}\right), \\
l^2 &= \frac{1}{2k^2 d_{12}}\left(2k\Gamma_{12}^2 - k^3\Gamma_{22}^1 - k\Gamma_{11}^1 + \frac{\partial k}{\partial u^1}\right).
\end{aligned} \tag{6.19}
$$

The *associate conjugate net* to the given net has the equation $k^2(du^1)^2 + (du^2)^2 = 0$. The l-parameters of the axis of the associate conjugate net, obtained by replacing k by ki ($i = \sqrt{-1}$) in (6.19), are given by

$$
\begin{aligned}
l^1 &= \frac{1}{2k^2 d_{12}}\left(2k^2\Gamma_{12}^1 + \Gamma_{11}^2 - k^2\Gamma_{22}^2 - k\frac{\partial k}{\partial u^2}\right), \\
l^2 &= \frac{1}{2k^2 d_{12}}\left(2k\Gamma_{12}^2 + k^3\Gamma_{22}^1 - k\Gamma_{11}^1 + \frac{\partial k}{\partial u^1}\right).
\end{aligned} \tag{6.20}
$$

Comparison of the l-parameters for the axes of a net (6.19) and its associate (6.20) reveals the fact that a necessary and sufficient condition for the two axes to be identical at a point is that $\Gamma_{22}^1 = \Gamma_{11}^2 = 0$. At such a point, moreover, the Darboux quadrics will have third order contact with S, as may be seen by reference to (2.26), the left member of which is the sum of the third order terms in the expansion just below (2.2), when asymptotics are parametric. Since third order contact is not possible at all points of a general surface, nor even at the points of a curve,[28] it is impossible to have the axis congruence of a conjugate net identical with that of its associate, except on a quadric surface. These observations are summarized in the theorem:

At points where the Darboux quadrics have third order contact with a surface, and only at such points, the axis of every conjugate net is identical with the axis of its associate net. This will hold at every point of a surface if and only if the surface is a quadric.

This theorem is proved projectively by Grove.[29]

It is known that the tangents to the curves of one family of a conjugate net, at points of a curve C_1 of the other family, generate a developable surface, namely, the surface enveloped by the tangent planes to the surface S along C_1. Eisenhart[30] uses this property as a definition of conjugate directions. The *ray* of the

[28] Lane, *op. cit.*, p. 77.

[29] V. G. Grove, *A General Theory of Surfaces and Conjugate Nets* (Transactions of the American Mathematical Society, Vol. 57, 1945), p. 116.

[30] Eisenhart, *op. cit.*, p. 231.

net at the point x^i is defined as the line joining the focal point on the C_1-tangent to the analogous focal point on the C_2-tangent.

The curve C_1 is given as before by $du^2 = kdu^1$, and its conjugate C_2 by $du^2 = -kdu^1$. A vector in the direction of the tangent to C_2 is $x^i_{,1} - kx^i_{,2}$. A general point on this tangent is

$$\xi^i = x^i + tx^i_{,1} - tkx^i_{,2}.$$

If ξ^i is a focal point, then as x^i traces C_1, $d\xi^i$ will be in the direction of $x^i_{,1} - kx^i_{,2}$, that is,

$$\begin{aligned} d\xi^i &= dx^i + tdx^i_{,1} + x^i_{,1}\,dt - tkdx^i_{,2} \\ &- tx^i_{,2}dk - kx^i_{,2}\,dt = m(x^i_{,1} - kx^i_{,2}). \end{aligned} \tag{6.21}$$

where m is a parameter. The relation (4.6) may be used to replace the second partial derivatives in (6.21). The result is multiplied by $X^i_{,1}$ and $X^i_{,2}$ successively, use being made of $X^i_{,\alpha}\, x^i_{,\beta} = -d_{\alpha\beta}$. If the parameter m is eliminated from the two equations so obtained, and kdu^1 is replaced by du^2, the resulting equation yields

$$t = \frac{2k}{\dfrac{\partial k}{\partial u^1} + k\dfrac{\partial k}{\partial u^2} - \Gamma^2_{11} + k^3\,\Gamma^1_{22} - k\,\Gamma^1_{11} + k^2\,\Gamma^2_{22}}. \tag{6.22}$$

To obtain the analogous parameter t' for the focal point of the tangent to C_1 as x^i moves along C_2, it is only necessary to replace k by $-k$ in (6.22). One obtains

$$t' = \frac{2k}{\dfrac{\partial k}{\partial u^1} - k\,\dfrac{\partial k}{\partial u^2} + \Gamma^2_{11} + \mathrm{k}^3\,\Gamma^1_{22} - \mathrm{k}\,\Gamma^1_{11} + \mathrm{k}^2\,\Gamma^2_{22}}. \tag{6.23}$$

The two focal points are respectively

$$x^i + t(x^i_{,1} - kx^i_{,2})$$

and

$$x^i + t'(x^i_{,1} + kx^i_{,2}),$$

and the ray of the net at x^i is given by

$$x^i + rt(x^i_{,1} - kx^i_{,2}) + (1 - r)t'(x^i_{,1} + kx^i_{,2}). \tag{6.24}$$

To find the points of intersection of the ray with the parametric tangents at x^i, one requires that the respective coefficients of $x^i_{,2}$ and $x^i_{,1}$ in (6.24) vanish; that is, for the point on $x^i_{,1}$,

$$-rtk + t'k - rt'k = 0,$$

$$r = \frac{t'}{t + t'} = \frac{\dfrac{1}{t}}{\dfrac{1}{t} + \dfrac{1}{t'}},$$

and for the point on $x^i_{,2}$,

$$rt + t' - rt' = 0,$$

$$r = \frac{t'}{t' - t} = \frac{\dfrac{1}{t}}{\dfrac{1}{t} - \dfrac{1}{t'}}.$$

From (6.22) and (6.23) it is found that

$$\frac{1}{t} + \frac{1}{t'} = \frac{1}{k}\left(\frac{\partial k}{\partial u^1} + k^3\,\Gamma^1_{22} - k\,\Gamma^1_{11}\right),$$

$$\frac{1}{t} - \frac{1}{t'} = \frac{1}{k}\left(k\frac{\partial k}{\partial u^2} - \Gamma^2_{11} + k^2\,\Gamma^2_{22}\right).$$

Use of the foregoing quantities in (6.24) yields, for the points of intersection of the ray with $x^i_{,1}$,

$$x^i + \frac{2k}{\dfrac{\partial k}{\partial u^1} + k^3\,\Gamma^1_{22} - k\,\Gamma^1_{11}}\,x^i_{,1}, \tag{6.25}$$

and for the intersection with $x^i_{,2}$

$$x^i + \frac{2k^2}{-\,k\dfrac{\partial k}{\partial u^2} + \Gamma^2_{11} - k^2\,\Gamma^2_{22}}\,x^i_{,2}. \tag{6.26}$$

In Section 3 it was shown that the points at which the reciprocal of any line with parameters l^s intersects the parametric tangents are given by

$$x^i + \frac{1}{l^s\,d_{s1} + S_1}\,x^i_{,1}$$

and

$$x^i + \frac{1}{l^s\,d_{s2} + S_2}\,x^i_{,2}.$$

With asymptotics parametric, these become

$$\begin{aligned} &x^i + \frac{1}{l^2\,d_{12} - \Gamma^2_{12}}\,x^i_{,1}, \\ &x^i + \frac{1}{l^1\,d_{12} - \Gamma^1_{12}}\,x^i_{,2}. \end{aligned} \tag{6.27}$$

If the parameters l^r of the associate net, given by (6.20), are used in (6.27), the results are identical with (6.25) and (6.26). This proves the theorem, given by Lane[31] for the projective case:

The ray congruence of a conjugate net is identical with the reciprocal of the axis congruence of the associate net.

[31] Lane, *op. cit.*, p. 97.

Since in all formulas the change from given net to associate net is made simply by replacing k by ki, it is evident that the ray congruence of the associate net is also the reciprocal of the axis congruence of the given net.

SECTION 7

THE CANONICAL PENCIL

One of the most interesting configurations in the projective theory[32] is the *canonical pencil*, some important lines of which are the *axis of Čech*, the *projective normal*, the *directrix of Wilczynski*, and the *edge of Green*. These four lines passing through a surface point x^i were originally defined independently by their geometric properties. In the words of Lane[32], "it is indeed remarkable that all of these lines . . . lie in a flat pencil." It is the principal purpose of this section to exhibit the l-parameters of the four lines, and to point out a metric theorem relating to three of them. The asymptotic curves will again be taken as the coordinate curves.

The parameters of the axis of Čech were given in (6.16), and will be repeated here for convenience:

$$\begin{aligned} l^1_{(a)} &= \frac{1}{2d_{12}}\left(\frac{1}{3}\frac{\partial}{\partial u^2}\log \Gamma^1_{22} - \frac{1}{3}\frac{\partial}{\partial u^2}\log \Gamma^2_{11} + 2\,\Gamma^1_{12} - \Gamma^2_{22}\right), \\ l^2_{(a)} &= \frac{1}{2d_{12}}\left(\frac{1}{3}\frac{\partial}{\partial u^1}\log \Gamma^2_{11} - \frac{1}{3}\frac{\partial}{\partial u^1}\log \Gamma^1_{22} + 2\,\Gamma^2_{12} - \Gamma^1_{11}\right), \end{aligned} \tag{7.1}$$

where the subscript (a) refers to the axis of Čech.

The projective normal is the only canonical line given in metric form by Lane.[33] Upon being changed to the present notation, its parameters are

$$l^1 = \frac{1}{2d_{12}}\left(\frac{\partial}{\partial u^2}\log \Gamma^1_{22} + \frac{\partial}{\partial u^2}\log \Gamma^2_{11} + \frac{1}{4g}\frac{\partial g}{\partial u^2} - \frac{3}{2d_{12}}\frac{\partial d_{12}}{\partial u^2}\right), \tag{7.2}$$

and a symmetrical expression for l^2. Use of the Codazzi relation

$$\frac{1}{d_{12}}\frac{\partial d_{12}}{\partial u^2} = \Gamma^2_{22} - \Gamma^1_{12},$$

and the identity[34]

$$\frac{1}{2g}\frac{\partial g}{\partial u^2} = \Gamma^1_{12} + \Gamma^2_{22}$$

permits (7.2) to be written in the form

$$\begin{aligned} l^1_{(n)} &= \frac{1}{2d_{12}}\left(\frac{\partial}{\partial u^2}\log \Gamma^1_{22} + \frac{\partial}{\partial u^2}\log \Gamma^2_{11} + 2\,\Gamma^1_{12} - \Gamma^2_{22}\right), \\ l^2_{(n)} &= \frac{1}{2d_{12}}\left(\frac{\partial}{\partial u^1}\log \Gamma^2_{11} + \frac{\partial}{\partial u^1}\log \Gamma^1_{22} + 2\,\Gamma^2_{12} - \Gamma^1_{11}\right), \end{aligned} \tag{7.3}$$

where the subscript (n) refers to the projective normal.

[32] Lane, *op. cit.*, p. 87.
[33] Lane, *op. cit.*, p. 243.
[34] Eisenhart, *op. cit.*, p. 150.

The directrix of Wilczynski at x^i may be defined as the line of the congruence for which the intersector net coincides with the reciprocal net of the congruence. Equations of these nets were given in (4.3) and (4.9). When expanded and reduced by use of $d_{11} = d_{22} = 0$, the second terms of these equations take the respective forms

$$(l^1_{,1} - l^2_{,2})du^1du^2, \tag{7.4}$$

and

$$(d^2_{12}l^2_{,2} - d^2_{12}l^1_{,1})du^1\,du^2. \tag{7.5}$$

Inspection of the coefficients of du^1du^2 in (7.4) and (7.5) shows that both nets are conjugate if $l^1_{,1} - l^2_{,2} = 0$. On the assumption that they are not conjugate, it is evident that the two equations (4.3) and (4.9) are identical if, and only if, the coefficients are proportional. The expressions (7.4) and (7.5) show clearly that the factor of proportionality must be $d \equiv -d^2_{12}$. If (4.3) is multiplied by $-d^2_{12}$ and its coefficients equated to those of (4.9), the resulting equalities are

$$\begin{aligned} d_{11}l^1 + (d_{11,2} - 2d_{12}S_1)l^2 &= -d^2_{12}g^{22} + S^2_1 - S_{1,1}\,, \\ (2d_{12}S_2 - d_{22,1})l^1 - d_{22,2}l^2 &= d^2_{12}g^{11} - S^2_2 + S_{2,2}\,, \end{aligned} \tag{7.6}$$

which, after appropriate reductions, give the solution

$$l^1 = \frac{1}{2d_{12}\,\Gamma^2_{11}}\left[d^2_{12}\,g^{22} - (\Gamma^2_{12})^2 - \frac{\partial}{\partial u^1}\,\Gamma^2_{12} + \Gamma^2_{12}\,\Gamma^1_{11} + \Gamma^1_{12}\,\Gamma^2_{11}\right], \tag{7.7}$$

and a symmetrical form for l^2. To transform the expression for l^1 into a more convenient form, use is made of the equalities $d = -d^2_{12} = R_{1212}$ and $R^h_{ijk} = g^{lh}R_{lijk}$, whence in particular, since $R_{2212} = 0$, it follows that

$$R^2_{121} = g^{22}R_{2121} = -g^{22}\,d^2_{12}\,.$$

Insertion of the expanded form[35] of the Riemann symbol R^2_{121} leads to the relation

$$-d^2_{12}\,g^{22} = -\frac{\partial}{\partial u^2}\,\Gamma^2_{11} - \frac{\partial}{\partial u^1}\,\Gamma^2_{12} + \Gamma^2_{11}\,\Gamma^2_{22} + \Gamma^1_{11}\,\Gamma^2_{12} - (\Gamma^2_{12})^2 - \Gamma^1_{12}\,\Gamma^2_{11}\,,$$

which may be substituted into (7.7). The result, together with the symmetrical development of l^2, is

$$\begin{aligned} l^1_{(d)} &= \frac{1}{2d_{12}}\left(-\frac{\partial}{\partial u^2}\log\Gamma^2_{11} + 2\,\Gamma^1_{12} - \Gamma^2_{22}\right), \\ l^2_{(d)} &= \frac{1}{2d_{12}}\left(-\frac{\partial}{\partial u^1}\log\Gamma^1_{22} + 2\,\Gamma^2_{12} - \Gamma^1_{11}\right), \end{aligned} \tag{7.8}$$

where the subscript (d) refers to the directrix.

It may be noted here that solutions for the directrix of Wilczynski may be calculated in general surface coordinates from (4.3) and (4.9). The details are

[35] Eisenhart, *op. cit.*, p. 100.

not included here, because the asymptotic coordinate system is more adaptable for the present purpose.

A condition for the coplanarity of three lines through x^i is readily shown to be

$$r l^{\alpha}_{(1)} + (1 - r) l^{\alpha}_{(2)} = l^{\alpha}_{(3)}, \qquad (\alpha = 1, 2)$$

where r is a parameter.

Inspection of (7.1), (7.3), (7.8) reveals the relationship

$$-\tfrac{1}{2} l^{\alpha}_{(n)} + \tfrac{3}{2} l^{\alpha}_{(a)} = l^{\alpha}_{(d)}, \qquad (\alpha = 1, 2) \tag{7.9}$$

which satisfies the coplanarity condition. It seems remarkable that the coefficients in (7.9) are constants, not functions of u^{α}, so that exactly the same relation holds at every point of the surface.

The *first canonical tangent* is the line in which the plane of the canonical pencil intersects the tangent plane to S at x^i. From (5.1) it is seen that the direction of this tangent is given by $(l^{\sigma}_{(1)} - l^{\sigma}_{(2)}) x^i_{,\sigma}$, where $l^{\sigma}_{(1)}$ and $l^{\sigma}_{(2)}$ refer to any two of the canonical lines. If the projective normal and the directrix of Wilczynski are used, the direction of the canonical tangent is given by

$$\begin{aligned} l^1_{(t)} &= \frac{1}{2d_{12}} \left(\frac{\partial}{\partial u^2} \log \Gamma^1_{22} + 2 \frac{\partial}{\partial u^2} \log \Gamma^2_{11} \right), \\ l^2_{(t)} &= \frac{1}{2d_{12}} \left(\frac{\partial}{\partial u^1} \log \Gamma^2_{11} + 2 \frac{\partial}{\partial u^1} \log \Gamma^1_{22} \right), \end{aligned} \tag{7.10}$$

where the subscript (t) refers to the tangent.

One should note here that since this tangent lies in the tangent plane, these parameters cannot be used to form an expression of the type $l^{\sigma} x^i_{,\sigma} + X^i$. The direction of the canonical tangent may be expressed by

$$\begin{aligned} l^1_{(t)} &= \frac{1}{2d_{12}} \frac{\partial}{\partial u^2} \log [\Gamma^1_{22} (\Gamma^2_{11})^2], \\ l^2_{(t)} &= \frac{1}{2d_{12}} \frac{\partial}{\partial u^1} \log [\Gamma^2_{11} (\Gamma^1_{22})^2]. \end{aligned} \tag{7.11}$$

One criterion for identifying the edge of Green is the fact[36] that it is the harmonic conjugate of the first canonical tangent with respect to the projective normal and the directrix of Wilczynski. This property is now used in order to write the l-parameters of the edge. For any line λ^i, the point Q given by $l^{\sigma} x^i_{,\sigma} + X^i$ lies in the plane parallel to the tangent plane and one unit above it, i.e. in the direction of the normal X^i. The projective normal, the directrix, and the edge intersect this plane in a range Q_n, Q_d, Q_e, upon which Q_t is evidently the point at infinity. Hence, by the harmonic property stated above, Q_e is the midpoint of the segment $Q_n Q_d$. If these three points are projected in the direction of X^i onto the tangent plane, and, in turn, if the points in the tangent plane are projected parallel to the parametric tangent $x^i_{,2}$ upon $x^i_{,1}$ the projection

[36] Lane, *op. cit.*, p. 93.

of Q_e is still the midpoint of the other two. Hence $l^1_{(e)}$ is the mean of $l^1_{(n)}$ and $l^1_{(d)}$. Similar considerations hold for $l^2_{(e)}$. Thus it follows that

$$l^{\alpha}_{(e)} = \tfrac{1}{2}\,{}^{\alpha}_{(n)} + \tfrac{1}{2}\,{}^{\alpha}_{(d)}, \qquad (\alpha = 1, 2)$$

the subscript (e) referring to the edge. From (7.3) and (7.8) one may write the l-parameters of the edge of Green:

$$(7.12) \qquad \begin{aligned} l^1_{(e)} &= \frac{1}{2d_{12}}\left(\frac{1}{2}\frac{\partial}{\partial u^2}\log \Gamma^1_{22} + 2\Gamma^1_{12} - \Gamma^2_{22}\right), \\ l^2_{(e)} &= \frac{1}{2d_{12}}\left(\frac{1}{2}\frac{\partial}{\partial u^1}\log \Gamma^2_{11} + 2\Gamma^2_{12} - \Gamma^1_{11}\right). \end{aligned}$$

The following metric theorem may be stated:

The edge of Green is a median of any triangle, two of whose sides are the projective normal and the directrix of Wilczynski, and whose third side is parallel to the tangent plane.

It follows as a corollary that if the edge of Green coincides with the metric normal at any point, then the projective normal and the directrix make equal angles with the metric normal.

The quantity $2\Gamma^1_{12} - \Gamma^2_{22}$, which appears in the expression for the parameter l^1 in each of the canonical lines studied above, can be written in the form

$$\frac{1}{4}\frac{\partial}{\partial u^2}\log g - \frac{3}{2}\frac{\partial}{\partial u^2}\log d_{12}$$

by use of the identities mentioned just above (7.3). An alternative form for the l-parameters of the projective normal, given in (7.3) is therefore

$$(7.13) \qquad l^{\alpha}_{(n)} = \frac{1}{2d_{12}}\frac{\partial}{\partial u^{\beta}}\log \frac{g^{1/4}}{d_{12}{}^{3/2}}\frac{\Gamma^{\alpha}_{\beta\beta}}{\Gamma^{\beta}_{\alpha\alpha}}. \qquad (\alpha \neq \beta)$$

Similar forms may be written for each of the other lines. Since the l-parameters of any line of the canonical pencil can be written

$$l^{\alpha} = rl^{\alpha}_{(n)} + (1 - r)l^{\alpha}_{(d)}, \qquad (\alpha = 1, 2)$$

it is readily seen from (7.3) and (7.8) that the general line of the pencil is given by

$$l^1 = \frac{1}{2d_{12}}\left[\frac{\partial}{\partial u^2}\log (\Gamma^1_{22})^r - \frac{\partial}{\partial u^2}\log (\Gamma^2_{11})^{2r-1} + 2\Gamma^1_{12} - \Gamma^2_{22}\right],$$

and the symmetric expression for l^2. In a form analogous to (7.13), the general canonical line is given by the parameters

$$(7.14) \qquad l^{\alpha} = \frac{1}{2d_{12}}\frac{\partial}{\partial u^{\beta}}\log \frac{g^{1/4}}{d_{12}{}^{3/2}}\frac{(\Gamma^{\alpha}_{\beta\beta})^r}{(\Gamma^{\beta}_{\alpha\alpha})^{2r-1}}, \qquad (\alpha \neq \beta)$$

where r may take any real value. As r takes the values $\frac{1}{3}$, $\frac{1}{2}$, 1, 0, the line coincides with the axis of Čech, the edge of Green, the projective normal, the directrix of Wilczynski, respectively.

Analytic conditions under which any canonical line coincides with the metric normal may easily be written from the expressions for the l-parameters. In the case of the axis of Čech, an interesting relation holds. Since the curves of Segre are union curves[37] with respect to the axis, a theorem follows immediately:

The axis of Čech coincides with the metric normal if, and only if, the curves of Segre are geodesics.

The reciprocals of the canonical lines intersect in a point y^i in the tangent plane, the polar point of the canonical plane. This polar point is called the *canonical point* corresponding to x^i. Its coordinates can be written from (5.3), using any two of the canonical lines as the lines $\lambda^i_{(1)}$ and $\lambda^i_{(2)}$. These coordinates, while easily written, are long expressions and will not be included here. The line joining x^i and y^i is called the *second canonical tangent.* From Section 5 it follows that the second canonical tangent is both conjugate and reciprocal to the first canonical tangent.

Section 8

RECIPROCAL CONIC OF A CONE OF LINES

If the line λ^i varies, always passing through the surface point x^i, its reciprocal line moves in the tangent plane. One may ask what will be the envelope of the reciprocal line for some restricted variation of λ^i. In the simplest case, in which λ^i may move in a plane, it has already been observed that the reciprocal line passes through a fixed point, as in the example of the canonical pencil treated in the last section. Another mode of variation of λ^i arises when λ^i generates a quadric cone. A discussion in synthetic terms will serve to make the situation clear before the analytic treatment is undertaken.

Let M be a conic in any plane π not containing the point x^i, and let the line λ^i move so as to trace M, generating a quadric cone C. By polar reciprocation with respect to the Darboux quadrics at x^i, the following transformations hold: the vertex x^i goes into the tangent plane τ; the plane π goes into a point Q; the points of M go into planes through Q, and the tangents to M into lines through Q, which are respectively the tangent planes and the elements of a cone M'. It is then evident that the reciprocal lines of λ^i envelop a conic C' on the plane τ, namely, the intersection of τ and the cone M'. The points of C' are the polars of the tangent planes of C. The conic C' will be called the *reciprocal conic* of the cone C generated by λ^i.

If a plane tangent to C passes through the center of a Darboux quadric, i.e., contains the line of centers, its polar point is at infinity, because it lies in the polar plane of the center, which is the plane at infinity. That is, if a plane tangent to C contains the line of centers, then the reciprocal conic has a point at infinity. Three cases therefore appear:

(1) When the line of centers is external to the cone C, two tangent planes will contain it; then the reciprocal conic has two real points at infinity and is a hyperbola.

[37] Springer, *U.C.*, p. 688.

(2) When the line of centers is an element of C, one tangent plane will contain it; then C' has one point at infinity and is a parabola.

(3) When the line of centers is internal to C, there is no tangent plane of C that contains it; then the conic C' has no real point at infinity, and is an ellipse. The following theorem may be stated:

When the line λ^i moves so as to pass through the points of a conic not coplanar with x^i, then the conic enveloped by the reciprocal lines will be a hyperbola, parabola, or ellipse according as the line of centers of the Darboux quadrics at x^i is external to, contained in, or internal to the cone generated by λ^i.

It is of some interest to determine the condition that the reciprocal conic C' pass through x^i, or that x^i be internal or external to C'. If x^i lies on C', then the polar plane τ of x^i is tangent to the cone C. If x^i is internal to C', then none of the tangents of C' passes through x^i, hence none of the lines λ^i lies on τ, the polar of x^i. If x^i is external to C', then two tangents of C' pass through x^i, so that two of the lines λ^i lie on τ. These observations are contained in the theorem:

The surface point x^i is internal to, lies on, or is external to the reciprocal conic according as the cone generated by λ^i is intersected by the tangent plane in one point, one line, or two lines.

For the analytic treatment, attention will be restricted to the case where λ makes a constant angle θ with the normal, generating a right circular cone. Since the surface point x^i is not to move in this study, it will be taken as the origin of a special orthogonal cartesian coordinate system, with z-axis normal to the surface, and with x- and y-axes tangent to the lines of curvature, which are also taken as coordinate curves on the surface.

The direction cosines of the tangent to the u^1-curve through x^i are $x^i_{,1}/\sqrt{g_{11}}$. A point designated by $x^i + Ax^i_{,1}$ can be written

$$x^i + \sqrt{g_{11}}A\left(\frac{x^i_{,1}}{\sqrt{g_{11}}}\right),$$

so that its distance from x^i is $\sqrt{g_{11}}A$. In the present local coordinate system its coordinates are $(\sqrt{g_{11}}A, 0, 0)$.

As developed in Section 3, the line reciprocal to λ^i crosses the parametric tangents at the points

$$x^i + \frac{1}{T_1}x^i_{,1}, \qquad x^i + \frac{1}{T_2}x^i_{,2}$$

respectively, where $T_\alpha = l^\sigma d_{\sigma\alpha} + S_\alpha$. The previous paragraph shows that the reciprocal line has intercepts $\sqrt{g_{11}}/T_1$ and $\sqrt{g_{22}}/T_2$ in the local coordinate system. Its equation in the xy-plane is therefore

$$\sqrt{g_{22}}T_1x + \sqrt{g_{11}}T_2y = \sqrt{g_{11}g_{22}} \tag{8.1}$$

It is geometrically evident that

$$g_{11}(l^1)^2 + g_{22}(l^2)^2 + 1 = \frac{1}{q^2} = \sec^2\theta,$$

whence

$$g_{11}(l^1)^2 + g_{22}(l^2)^2 = \tan^2\theta. \tag{8.2}$$

(It is recalled from (6.8) that in general coordinates, $g_{\alpha\beta}l^\alpha l^\beta = \tan^2\theta$). It follows from (8.2) that

$$l^2 = \frac{1}{\sqrt{g_{22}}}\sqrt{\tan^2\theta - g_{11}(l^1)^2}.$$

It is now possible to express (8.1) in terms of a single parameter l^1 and find the envelope of the line. With lines of curvature parametric, one has

$$T_1 = l^1 d_{11} + S_1, \qquad T_2 = l^2 d_{22} + S_2$$

whence (8.1) becomes

$$\sqrt{g_{22}}\,l^1 d_{11}\,x + \sqrt{g_{22}}\,S_1 x + \sqrt{\frac{g_{11}}{g_{22}}}\,d_{22}\sqrt{\tan^2\theta - g_{11}(l^1)^2}\,y + \sqrt{g_{11}}\,S_2 y - \sqrt{g_{11}g_{22}} = 0.$$

The usual method of eliminating the parameter to find the envelope yields the equation of the reciprocal conic,

$$\begin{aligned}(g_{22}^2 d_{11}^2\tan^2\theta - g_{22}^2 g_{11}S_1^2)x^2 - (2g_{11}^{\frac{3}{2}}g_{22}^{\frac{3}{2}}S_1S_2)xy \\ + (g_{11}^2 d_{22}^2\tan^2\theta - g_{11}^2 g_{22}S_2^2)y^2 + (2g_{22}^2 g_{11}^{\frac{3}{2}}S_1)x + (2g_{11}^2 g_{22}^{\frac{3}{2}}S_2)y - g_{11}^2 g_{22}^2 = 0.\end{aligned} \tag{8.3}$$

It appears now that the origin x^i will be the center of the reciprocal conic if and only if $S_1 = S_2 = 0$, i.e., $\frac{\partial K}{\partial u^\alpha} = 0$, $(\alpha = 1, 2)$, at the point, where K is the Gaussian curvature of the surface at x^i. If K is constant for the surface, then every surface point will be the center of the corresponding reciprocal conic.

From (8.3) it is possible to derive conditions that the reciprocal conic be a hyperbola, a parabola, or an ellipse. By setting the discriminant equal to zero, one finds the condition

$$\tan\theta = \frac{1}{d_{11}d_{22}}\sqrt{g_{22}\,d_{11}^2 S_2^2 + g_{11}\,d_{22}^2 S_1^2}\,. \tag{8.4}$$

For the line of centers, (2.31) yields, in the present coordinate system, the parameters

$$l^1 = -d^{11}S_1, \qquad l^2 = -d^{22}S_2\,.$$

If θ_c is the angle between the line of centers and the normal, (8.2) yields after a short calculation

$$\tan\theta_c = \frac{1}{d_{11}d_{22}}\sqrt{g_{11}\,d_{22}^2 S_1^2 + g_{22}\,d_{11}^2 S_2^2}\,. \tag{8.5}$$

Comparison of (8.4) and (8.5) shows that the reciprocal conic is a parabola, if and only if $\theta = \theta_c$. If θ is greater or less than this critical value, the discriminant

is negative or positive. This completes an analytic proof of the first theorem of this section, for the case where θ is constant.

If $\dfrac{\partial K}{\partial u^\alpha} = 0$, x^i is the center of the reciprocal conic, which in this case is an ellipse because the line of centers is normal to the surface and hence is internal to the cone generated by λ^i. The equation of this ellipse is found, by setting $S_1 = S_2 = 0$ in (8.3), to be

$$(g_{22}^2 d_{11}^2 \tan^2\theta)x^2 + (g_{11}^2 d_{22}^2 \tan^2\theta)y^2 = g_{11}^2 g_{22}^2 .$$

By use of the expressions

$$\rho_1 = \frac{d_{11}}{g_{11}}, \qquad \rho_2 = \frac{d_{22}}{g_{22}}$$

for the principal radii of normal curvature[38] of the surface at x^i, the equation of the ellipse reduces to

$$\frac{x^2}{\left(\dfrac{1}{\rho_1}\cot\theta\right)^2} + \frac{y^2}{\left(\dfrac{1}{\rho_2}\cot\theta\right)^2} = 1. \tag{8.6}$$

This ellipse has principal axes tangent to the lines of curvature, and intercepts on those axes proportional to the reciprocals of the principal radii of normal curvature, the factor of proportionality being the cotangent of the generating angle θ. The major axis is in the direction having the smaller radius of normal curvature. For $\theta = \dfrac{\pi}{4}$, the intercepts are exactly $\dfrac{1}{\rho_1}$ and $\dfrac{1}{\rho_2}$. The eccentricity, independent of θ, is given by

$$e^2 = 1 - \frac{\rho_1}{\rho_2} \quad \text{or} \quad e^2 = 1 - \frac{\rho_2}{\rho_1},$$

according as ρ_2 is greater or less than ρ_1. The ellipse becomes a circle if $\rho_1 = \rho_2$, which means that x^i is a spherical or umbilical point on S. The ellipse (8.6) is thus an indicatrix for the behavior of the surface at the point. The preceding observations are summarized in the following theorem:

At all points of a surface of constant Gaussian curvature, or at a point of any surface where $\dfrac{\partial K}{\partial u^\alpha} = 0$, *the reciprocal conic induced when* λ^i *generates a right circular cone with axis normal to the surface, is an ellipse whose center is the surface point, whose axes coincide with the directions of the lines of curvature, and whose eccentricity is given by* $e^2 = 1 - \dfrac{\rho_1}{\rho_2}$ $(\rho_2 > \rho_1)$. *The ellipse is a circle at a spherical point of the surface.*

In the more general case where $\dfrac{\partial K}{\partial u^\alpha} \neq 0$, it develops that the angle α of inclina-

[38] Eisenhart, *op. cit.*, p. 239.

tion of the axis of the reciprocal conic (8.3) is related to the geometry of the surface in several ways. By the usual procedure of analytic geometry it is found that

$$\tan 2\alpha = \frac{-2g_{11}^{\frac{1}{2}} g_{22}^{\frac{1}{2}} S_1 S_2}{\tan^2 \theta (g_{22}^2 d_{11}^2 - g_{11}^2 d_{22}^2) - g_{11} g_{22} (g_{22} S_1^2 - g_{11} S_2^2)}. \tag{8.7}$$

From this expression it is seen that $\tan 2\alpha$ decreases toward zero in absolute value as θ approaches $\frac{\pi}{2}$; the inclination therefore approaches either $\frac{\pi}{2}$ or zero, whence there follows the result:

The axes of the reciprocal conic, for λ^i generating a right circular cone, approach coincidence with the lines of curvature as the generating angle θ approaches $\frac{\pi}{2}$.

When $\theta = \theta_c$, from (8.5), the denominator of (8.7) becomes

$$\frac{g_{22}^3 d_{11}^4 S_2^2 - g_{11}^3 d_{22}^4 S_1^2}{d_{11}^2 d_{22}^2}$$

so that for $\theta = \theta_c$, when the conic is a parabola, one has the relation

$$\tan 2\alpha_c = \frac{2g_{11}^{\frac{3}{2}} g_{22}^{\frac{3}{2}} d_{11}^2 d_{22}^2 S_1 S_2}{g_{11}^3 d_{22}^4 S_1^2 - g_{22}^3 d_{11}^4 S_2^2},$$

whence by inspection one obtains the inclination of the axis of the parabola:

$$\tan \alpha_c = \frac{g_{22}^{\frac{3}{2}} d_{11}^2 S_2}{g_{11}^{\frac{3}{2}} d_{22}^2 S_1} = \frac{\sqrt{g_{22}}\, \rho_1 d_{11} S_2}{\sqrt{g_{11}}\, \rho_2 d_{22} S_1}. \tag{8.8}$$

Now the plane of the normal and the line of centers makes an angle with the x-axis given by

$$\tan \beta = \frac{\sqrt{g_{22}}\, d_{11} S_2}{\sqrt{g_{11}}\, d_{22} S_1},$$

as can be seen from the values of l^α in (2.31). This angle equals α_c when x^i is a spherical point. By rotating the axis through an angle α_c it is found that x^i lies on the axis of the parabola if x^i is a spherical point, so that the plane of the normal and the line of centers intersects the tangent plane in this axis.

If λ^i varies so as to pass through the points of a skew curve instead of the conic studied above, the polars of the tangents to that curve generate a ruled surface, whose intersection with the tangent plane τ is the reciprocal curve of the cone generated by λ^i. This more general case has not been investigated further.

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DENISON SCIENTIFIC ASSOCIATION

Organized April 16, 1887

PROGRAM OF THE 1946–1947 SCHOOL YEAR. A REPORT BY THE PERMANENT SECRETARY

The following officers served the Denison Scientific Association during 1946–1947.

George D. Morgan, Department of Biology, *President*
Frank J. Wright, Department of Geology, *Vice-President*
Edson C. Rupp, Department of Mathematics, *Secretary-Treasurer*
Leon E. Smith, Department of Physics, *Librarian*
W. Clarence Ebaugh, Emeritus Professor of Chemistry, *Editor*
W. Alfred Everhart, Department of Chemistry, *Permanent Secretary and Editor*

On January 1, 1947 Dr. W. C. Ebaugh retired from active duty as Editor of the *Journal of the Scientific Laboratories*. At a business session of the Association, date of November 26, 1946, Dr. W. A. Everhart, Associate Professor of Chemistry, was appointed to the editorship of the Journal and was asked to continue in his duties as Permanent Secretary.

The meetings of the year were held in the auditorium of the Life Science Building at Denison University, were open to public admission, were directed and presented by the President, Dr. Morgan. A brief description of the program that was carried out follows.

September 22, 1946

NATIONAL STEREOTYPES. Frederick G. Detweiler

The speaker questioned the truth implied in such common general statements or premises as,—that once a Jap always a Jap, that one can never trust a Greek, that the Poles are born to hate the Russians, and that the English always muddle through. There are such qualities as general national characteristics. But they do not apply to all members of national groups. And the acceptances of such generalities as are implied in the type of premises mentioned often serve as the bases of misunderstandings between nations.

October 8, 1946

STREAM PIRACIES IN THE SOUTHERN APPALACHIANS. Frank J. Wright

Streams of water are competitors. When one stream has an advantage in erosive power over one of its neighbors it may cut down the divide between them

and abstract the water from the less vigorous stream. Many such changes have occurred in the Appalachians, both northern and southern.

Since the erosional history of a region is intimately connected with stream histories, it is important to trace the latter in order to reconstruct the former. An understanding of past events is essential to the correct interpretation of present topography.

Classical examples of stream piracy in the southern Appalachians were mentioned and discussed, with reference to a relief map. But chief attention was given to cases which have been described in more recent studies. And a working hypothesis for the explanation of widely distributed piracies was suggested.

October 22, 1946

MOTION PICTURES OF THE RESULTS OF THE BOMBING OF HIROSHIMA AND NAGASAKI. Introductory remarks by STUART STURGIS.

The film used in showing the bombing of Hiroshima and Nagasaki, Japan,—both technique and effects,—was an army film in technicolor (in part) which had not yet been released for general public use.

Preceding the showing of the film Dr. Stuart Sturgis, introduced by Dr. Albert W. Davison, research director of the Owens-Corning Fiberglas Corporation of Newark, Ohio, spoke briefly on certain phases of atomic bomb characteristics. Dr. Sturgis had been a supervisor of several districts that participated in the manufacture of chemicals used in the type of bombs employed in Japan. And he was an eye-witness to the recent atomic bomb tests at Bikini Island.

November 12, 1946

ROADSIDE GLIMPSES OF MATHEMATICS. FORBES B. WILEY

One may take a trip, to illustrate, from Chicago to New York, with blinds drawn or with some other device to render himself entirely oblivious to all interesting way-side sights and experiences incidental to such a trip. Such an individual may succeed in reaching his destination. But he arrives there without the horizon widening experiences which could have been his while enroute.

An analagous situation may arise when a student pursues a course in mathematics. And the speaker explained how he has experimented for some time with an endeavor to conduct mathematics courses, particularly freshman classes in mathematics, along a *scenic* route, with short "stop-overs" at exceptionally interesting points. The talk described some of these road-side scenes.

November 26, 1946

OUR NATION'S HEALTH LIES IN THE SOIL. OLLIE E. FINK (National Secretary, *Friends of the Land*)

The health of a nation depends directly upon the fertility of its soil, which in turn depends upon a proper distribution and abundance of minerals and water. This is the reason for the emphasis placed on soil conservation in our country.

And conservation is as important for city dwellers as it is for farmers. In fact it is a problem of primary concern to the city dwellers. In Revolutionary War

times it required the work of about nineteen persons to raise enough food to supply twenty persons. And thus there was but little surplus for city supplies. Now, with methods and machinery that are available, one person can produce enough food for ten, if the soil is kept fertile. But whenever the farmer for any reason ceases to produce a surplus, the urban population is forced to return to rural life and to agriculture.

In the United States of America we are at the present time relatively well off in that we have about three and one half acres of arable land available per person in our population. But in the world as a whole the people are less fortunate since there is not enough tillable land to provide as much as two acres per person. And that is why, at least it is one reason why, twenty to thirty million people die annually from malnutrition and hunger.

In considering conservation it is well to note particularly the fundamental importance of water and its proper distribution. Really, we live upon trapped rain drops. And the progress of civilization depends upon them. Numerous examples were cited to illustrate these facts and to prove that water is a vital factor in both vegetable and animal life.

December 10, 1946

APTITUDE TESTING AND ITS PLACE IN VOCATIONAL GUIDANCE. Robert E. Dixon

The presentation included a discussion of the concept of aptitudes, methods now in common use for measuring aptitudes, and the usefulness of the results of such measurements in the practice of vocational counseling.

Typical tests were mentioned and discussed briefly. And a prediction of new emphases and possible future trends in the use and application of such tests was stated.

January 14, 1947

MORALE VERSUS MATERIEL. Conrad E. Ronneberg

The story of the development of modern weapons of war is not new or untold. Submarines, air-craft carriers, automatic anti-aircraft artillery, radar, sonar, the proximity fuse, the atomic bomb,—these and many other scientific devices constitute the materiel of modern war. And these our nation produced in such quantities in the recent war years as to cause us to be called—"the arsenal of democracy".

But there is another aspect of warfare, that which involves the morale of men, which has not been so well publicized. And Napoleon rated the value of morale to materiel as of the order of three to one. Victory in World War II was in large part the result of an unusually high state of morale among our allied soldiers and sailors. Materiel was important. But the Maginot Line did not save France. And a superior fleet in the Mediterranean Sea did not mean victory for the Italians.

The methods used to achieve this high state of morale among American military personnel, methods made possible because our army and navy were the

instruments of a free society, were listed and discussed in considerable detail, then illustrated in part by the use of a motion picture film, entitled "Follow Me Again", an educational film (EF-3) produced by the War Department.

February 11, 1947

Meeting omitted.

February 25, 1947

WHERE DO WE STAND WITH CANCER? RALPH E. PICKETT, M.D.

Introductory discussion included a clarification of the nature of cancers and of their causative factors. There followed a brief history of the progress of cancer research up to the present time, and mention of various trends toward future research in this field. And the address was concluded with a discussion of the biology of cancer cells, diagnostic tests, and the generally accepted modern methods of treatment.

This address, together with a very full bibliography on the cancer problem, was published in the *Journal of the Scientific Laboratories*, Volume 40, Article 3, April 1947.

March 11, 1947

A TRIP WITH A BOTANIST. RICHARD V. MORRISSEY

A lecture, illustrated by the use of many colored slides, which dealt with . . . first, several types of plantings, both wild and cultivated; second, some types of hardy perennial plants; third, some of the unusual and colorful flora of the Denison campus.

March 25, 1947

GEOGRAPHICAL INFLUENCES IN THE HISTORY OF GRANVILLE, MASSACHUSETTS. MRS. DOROTHY V. N. BROOKS

In 1800 the town of Granville, Massachusetts, had a population of about two thousand five hundred; and it was thus comparable in size to present day Granville, Ohio. It was a thriving community with a vitality based on the natural resources of the region.

Now, although the area on which the town stands, overlooking the Connecticut valley, has had continuous occupation since it was first purchased from the Indians in 1686, for one gun and sixteen brass buttons, this scattered community in the Berkshires numbers only some six hundred residents.

Much of the beauty and attractiveness of the site still exists. But industries and homes have been moved away to more desirable places in response to changing demands (for local material), to inventions which have altered transportation, to expansion of manufacturing and agriculture in the Connecticut valley, to changing conditions in all of Massachusetts and in the nation. The whole nature of this once bustling village has altered, in direct reflection of geographic influences operating in successive stages of its history.

April 8, 1947

RECENT ADVANCES IN COSMIC RAYS. EDWARD W. DEEDS

Important facts concerning the nature of cosmic rays were presented in an historical and elementary fashion, in popular style rather than in the technical language of the physicst and mathematician. Illustrations and demonstrations, including the use of a Geiger counter, were employed. And an emphasis was placed upon the major role played by cosmic ray researches in bringing about our modern concepts of atomic nuclear processes and changes.

April 22, 1947

PROJECTIVE TECHNIQUES OF PSYCHOLOGY. MRS. ELIZABETH C. STRICKLAND

A discussion of techniques used by psychologists in evaluating individual intelligence and personality patterns. Older methods were compared with the newer, with special emphasis on projective techniques. And some of the newer tests were described, then illustrated by the use of a motion picture film depicting studies of child behavior.

May 13, 1947

Dinner and Business Meeting at the Granville Inn.

Annual meeting, attended by members, only.

Items of business concerned primarily the budget of the *Journal*, care of the accumulating library of scientific journals received in exchange, election of officers for the year 1947–1948.

The result of the election was as follows.

President, R. H. HOWE, Department of Physics
Vice-President, A. W. LINDSEY, Department of Biology
Secretary-Treasurer, W. C. BIEL, Department of Psychology
Librarian, L. E. SMITH, Department of Physics
Permanent Secretary and Editor of the JOURNAL, W. A. EVERHART, Department of Chemistry

Messages were ordered sent to the following members. To Dr. Frank J. Wright, greetings and best wishes for a speedy recovery from illness suffered while on leave of absence during the spring months. To Dr. Arthur W. Lindsey, congratulations because of his recent election to the Presidency of the Ohio Academy of Science.

Recognition and congratulations were expressed, also, to the following members who have received election to executive committee memberships in sections of the Ohio Academy of Science. Dr. Lynde C. Steckle, Psychology; Dr. Frederick G. Detweiler, Sociology; Dr. W. Alfred Everhart, Chemistry.

Respectfully submitted,

W. A. EVERHART,
Permanent Secretary